# Toposes of presheaves on a monoid and their points

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# Toposes

A Grothendieck topos is a category of sheaves $\mathbf{Sh}(\mathcal{C}, J)$ where $\mathcal{C}$ is a small category and $J$ is a Grothendieck topology on $\mathcal{C}$.

Examples:

- the category of sets;
- $\mathbf{Sh}(X)$ for $X$ a topological space;
- the petit étale topos $\mathbf{Sh}(X_{\text{ét}})$ for $X$ a scheme;
- the category of $G$-sets for $G$ a group;
- the category of directed graphs.

# Properties of toposes

Toposes have all colimits and limits, and

$$\operatorname*{colim}_{i \in I} (X_i \times_Z Y) \simeq \left( \operatorname*{colim}_{i \in I} X_i \right) \times_Z Y.$$

*Idea: gluing things together is stable under base change*

Every morphism in a topos can be written in a unique way as an epimorphism followed by a monomorphism.

# Geometric interpretation

We often identify a topological space $X$ with the topos $\mathbf{Sh}(X)$.

The objects of a more general topos $\mathcal{E}$ are then thought of as "sheaves on $\mathcal{E}$", and in this way $\mathcal{E}$ becomes a generalized topological space.

If $X$ is a topological space equipped with a continuous $G$-action, for a discrete group $G$, then there is a topos $\mathbf{Sh}_G(X)$ of $G$-equivariant sheaves on $X$. This is the "formal/stacky quotient" of $X$ by $G$, as opposed to the coarse quotient $\mathbf{Sh}(G\backslash X)$.

# Special case: presheaf toposes

A presheaf topos is a category of the form

$$\mathbf{PSh}(\mathcal{C}) \simeq [\mathcal{C}^{\mathrm{op}}, \mathbf{Sets}]$$

for $\mathcal{C}$ a small category.

A monoid is a small category with one object. We will consider categories of the form $\mathbf{PSh}(M)$ for $M$ a monoid.

# Geometric morphisms

A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is an adjunction

$$\mathcal{F} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathcal{E}$$

with $f^*$ preserving finite limits.

Further, $f$ is called essential if $f^*$ has a further left adjoint $f_!$.

# Geometric morphisms

This is analogous to topology: in topology each continuous map $f : Y \to X$ gives an adjunction

$$\mathcal{O}(Y) \overset{f^{-1}}{\leftrightarrows} \mathcal{O}(X)$$

with $f^{-1}$ preserving finite limits.

# The base topos **Sets**

Each (Grothendieck) topos $\mathcal{E}$ has a unique geometric morphism

$$\mathcal{E} \xrightarrow{\gamma} \textbf{Sets}$$

with

- $\gamma^*(S)$ the constant sheaf on $S$; and
- $\gamma_*(E) = \Gamma(E) = \mathrm{Hom}_{\mathcal{E}}(1, E)$ the global sections of $E$.

We call $\gamma$ the global sections geometric morphism.

# Points

A point of a topos $\mathcal{E}$ is a geometric morphism $p : \mathbf{Sets} \to \mathcal{E}$.

Here $E_p = p^*(E)$ is often called the stalk of $E$ at $p$,
and $p_*(S)$ is the skyscraper sheaf on $S$ at $p$.

A morphism of points $p \to q$ is a natural transformation $p^* \Rightarrow q^*$.
The category of points of $\mathcal{E}$ is denoted by $\mathbf{Pts}(\mathcal{E})$.

# Points of topological spaces

**Proposition**

*Let $X$ be a topological space. There is a bijection between isomorphism classes of points of $\mathbf{Sh}(X)$ and irreducible closed subsets of $X$.*

The closure of a singleton $\overline{\{x\}}$ is always irreducible and closed.

We say that $X$ is sober if every irreducible closed subset is the closure of a unique point. For sober spaces, there is a bijection between points of $\mathbf{Sh}(X)$ and elements of $X$.

# Sober spaces

Examples:

- any Hausdorff space;
- the spectrum of a ring with the Zariski topology;
- the line with double origin;
- the Sierpiński space.

Non-examples:

- an infinite set with the indiscrete topology;
- an infinite set with the cofinite topology.

# Sobrification

Each topological space $X$ has a sobrification $\hat{X}$. This is a sober space such that

$$\mathbf{Sh}(X) \simeq \mathbf{Sh}(\hat{X}).$$

So points of $\mathbf{Sh}(X)$ are given by elements of $\hat{X}$.

# Monoid toposes

Let $M$ be a monoid. What are the points of $\mathbf{PSh}(M)$?

Note that $\mathbf{PSh}(M)$ has an alternative description as the category of right $M$-sets.

# Motivation

The Arithmetic Site by Connes and Consani is the topos $\mathbf{PSh}(\mathbb{N}_+^\times)$, for $\mathbb{N}_+^\times = \{1, 2, 3, \dots\}$, seen as monoid under multiplication, equipped with a sheaf of semirings as structure sheaf.

**Theorem (Connes–Consani)**

*The points of* $\mathbf{PSh}(\mathbb{N}_+^\times)$ *are classified up to isomorphism by the double quotient*

$$\mathbb{Q}_+^* \backslash \mathbb{A}^f / \widehat{\mathbb{Z}}^*.$$

# Motivation

**Questions:**

- The monoid $\mathbb{N}_+^\times$ is a free commutative monoid on the primes. Why do we still get uncountably many points?
- Is the relation to the adeles a coincidence?

# Motivation

**Theorem (Sagnier)**

*Let $K$ be an imaginary quadratic field with class number $1$, and let $M$ be the monoid of nonzero elements in the ring of integers $\mathcal{O}_K \subset K$. Then the points of $\mathbf{PSh}(M)$ are classified up to isomorphism by the double quotient*

$$K^*\backslash\mathbb{A}_K^f/\widehat{\mathcal{O}_K}^*.$$

Alternatively, the points of $\mathbf{PSh}(M)$ correspond to nonzero $\mathcal{O}_K$-submodules of $K$.

# Flat functors

If $\mathcal{C}$ is a small category, then points of $\mathbf{PSh}(\mathcal{C})$ correspond to functors

$$F : \mathcal{C} \to \mathbf{Sets}.$$

that are flat in the sense that

- $F(c) \neq \varnothing$ for some $c$ in $C$;
- if $x \in F(c)$ and $y \in F(c')$ then there is a diagram $c \leftarrow d \to c'$ and an element $z \in F(d)$ such that $z|_c = x$ and $z|_{c'} = y$;
- for two morphisms $f, g : d \to c$ and $x \in F(d)$ with $x|_f = x|_g$ there is a morphism $h : e \to d$ with $fh = gh$ and $z \in F(e)$ such that $z|_h = x$.

# Flat left $M$-sets

If $M$ is a monoid, then points of $\mathbf{PSh}(M)$ correspond to *left $M$-sets* $A$ that are flat in the sense that

- (non-empty) $A \neq \varnothing$;
- (locally cyclic) if $x, y \in A$ then there are elements $m, m' \in M$ and an element $z \in A$ such that $mz = x$ and $m'z = y$.
- (condition (E)) for two elements $m, m' \in M$ and an element $a \in A$ such that $ma = m'a$, there is an element $m'' \in M$ and an element $b \in A$ such that $mm'' = m'm''$ and $a = m''b$.

Further, $A$ is flat if and only if $- \otimes_M A$ preserves finite limits.

# Exercise

Let $X$ and $Y$ be infinite sets with $|X| \leq |Y|$.

Prove that $\mathrm{Hom}(X, Y)$ is flat as right $\mathrm{End}(X)$-set.

This means that the category of points of **PSh**$(\mathrm{End}(X)^{\mathrm{op}})$ is not small!

# Case of the Arithmetic Site

*(From Connes and Consani, "Geometry of the Arithmetic Site".)*

Let $A$ be a flat left $\mathbb{N}_+^\times$-set. Then for two elements $x, y \in A$ we can find elements $m, m' \in \mathbb{N}_+^\times$ and an element $z \in A$ such that $x = mz$ and $y = m'z$.

Now we define:

$$x + y = (m + m')z.$$

Connes and Consani then show that $(A, +)$ is isomorphic to $L \cap \mathbb{Q}_+$ for $L \subseteq \mathbb{Q}$ a nonzero subgroup.

# Ind-categories

The category of points of $\mathbf{PSh}(\mathcal{C})$ is also equivalent to $\mathbf{Ind}(\mathcal{C}^{\mathrm{op}})$, the category of formal filtered colimits, with as morphisms

$$\mathrm{Hom}_{\mathbf{Ind}(\mathcal{C}^{\mathrm{op}})}(\varinjlim_{i\in I} X_i, \varinjlim_{j\in J} Y_j) \simeq \varprojlim_{i\in I} \varinjlim_{j\in J} \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X_i, Y_j).$$

If $\mathcal{A}$ is a locally finitely presentable category, then $\mathcal{A} \simeq \mathbf{Ind}(\mathcal{A}_{\mathrm{fp}})$.

# Ind-categories

Examples:

- the category of points for $\mathbf{PSh}(\mathbf{Comm}_{\mathrm{fp}}^{\mathrm{op}})$ is the category of rings;
- the category of points for $\mathbf{PSh}(\mathbf{FinSets}^{\mathrm{op}})$ is the category of sets.

How can we apply this idea to $\mathbf{PSh}(M)$ for $M$ a monoid?

# Ind-categories

Let $M = (\mathbb{Z}, \cdot)$ the monoid of integers under multiplication.

Then $M \cong \mathrm{End}_{\mathbf{Ab}}(\mathbb{Z})$ for **Ab** the category of abelian groups (which is lfp, with $\mathbb{Z}$ finitely presented).

So the category of points of $\mathbf{PSh}(M)$ is the full subcategory of **Ab** consisting of those abelian groups that are filtered colimits of copies of $\mathbb{Z}$.

These are precisely the subgroups of $\mathbb{Q}$.

# Ind-categories

| Monoid $M$ | Category of points $\mathcal{L}$ | Subcategory $\mathcal{L}^{\mathrm{fp}} \subseteq \mathcal{L}$ |
|---|---|---|
| $\mathbb{N}_+^\times$ | Nontrivial ordered subgroups of $\mathbb{Q}$, injective morphisms of ordered groups | Ordered subgroups isomorphic to $\mathbb{Z}$ |
| $\mathbb{N}_0^\times$ | Ordered subgroups of $\mathbb{Q}$, morphisms of ordered groups | Ordered subgroups isomorphic to $\mathbb{Z}$ |
| $\mathbb{Z}_\pm = \{z \in \mathbb{Z} : z \neq 0\}$ | Nontrivial subgroups of $\mathbb{Q}$, injective group morphisms | Subgroups isomorphic to $\mathbb{Z}$ |
| $\mathbb{Z}$ | Subgroups of $\mathbb{Q}$, group morphisms | Subgroups isomorphic to $\mathbb{Z}$ |
| $\mathrm{M}_n^{\mathrm{ns}}(\mathbb{Z}) = \{a \in \mathrm{M}_n(\mathbb{Z}) : \det(a) \neq 0\}$ | Subgroups $A \subseteq \mathbb{Q}^n$ such that $A \otimes \mathbb{Q} \cong \mathbb{Q}^n$, injective group morphisms | Subgroups isomorphic to $\mathbb{Z}^n$ |
| $\mathrm{M}_n(\mathbb{Z})$ | Subgroups $A \subseteq \mathbb{Q}^n$, group morphisms | Subgroups isomorphic to $\mathbb{Z}^n$ |

# Equivariant sheaves

Let $X$ be a topological space equipped with a continuous $G$-action, for $G$ a discrete group.

Suppose that $X$ has a basis of open sets $\mathcal{B}$ on which $G$ acts transitively.

Then $\mathbf{Sh}_G(X) \simeq \mathbf{Sh}_G(\mathcal{B}) \simeq \mathbf{Sh}(M, J)$ for

$$M = \{g \in G : g(U) \subseteq U\}$$

and a certain Grothendieck topology $J$.

# Equivariant sheaves

So $\mathbf{Sh}_G(X) \simeq \mathbf{Sh}(M, J)$. When is $J$ the presheaf topology?

This is precisely when

$$U = \bigcup_{i \in I} U_i \;\Rightarrow\; \exists i \in I,\; U = U_i$$

for open sets $U, U_i$ in $\mathcal{B}$. We say that $\mathcal{B}$ is a minimal basis.

In this case, $\mathbf{Sh}_G(X) \simeq \mathbf{PSh}(M)$. The points of $\mathbf{PSh}(M)$ are then given by elements $x \in \hat{X}$. Morphisms of points $x \to y$ are elements $g \in G$ such that $x \leq g(y)$.

# Equivariant sheaves

Conversely, suppose that $G$ is a group and $M \subseteq G$ a submonoid.

Take $X = G/M^*$ with as basis of open sets $\mathcal{B}$ the sets of the form

$$U_g = \{gm : m \in M\}.$$

The left $G$-action on $G/M^*$ by multiplication is continuous and works transitively on $\mathcal{B}$.

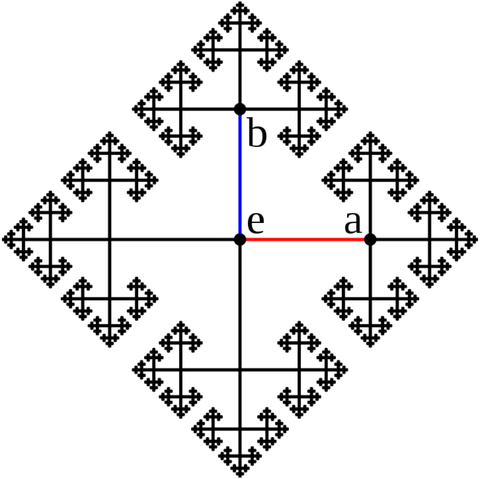
So $\mathbf{PSh}(M) \simeq \mathbf{Sh}_G(X)$.

# Equivariant sheaves

Example: the free monoid $M = \langle a_1, \ldots, a_n \rangle$.

We consider $M$ as subgroup of the free group $G$ on $n$ generators.
In this case, $M^* = 1$, so $X = G/M^* = G$.

# Equivariant sheaves



# Equivariant sheaves

Conclusion: the points of $\mathbf{PSh}(M)$ correspond to possibly infinite words

$$x = g x_1 x_2 x_3 \dots$$

with each $x_i \in \{a_1, \dots, a_n\}$ and $g \in G$.

Further, we can compute endomorphism monoids of these points:

- $\mathrm{End}(x) \cong M$ if $x$ is a finite word;
- $\mathrm{End}(x) \cong \mathbb{Z}$ if $x$ is eventually periodic;
- $\mathrm{End}(x) \cong 1$ in all other cases.

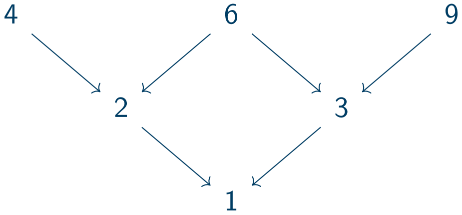
# Equivariant sheaves

Example: the monoid $\mathbb{N}_+^\times$ of nonzero natural numbers under multiplication.

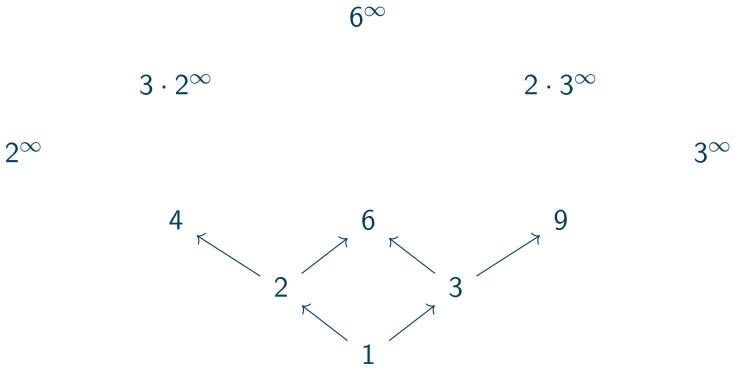
We take $\mathbb{N}_+^\times \subset \mathbb{Q}_+^*$. Again, $1 \in \mathbb{N}_+^\times$ is the only invertible element.
So $X = \mathbb{Q}_+^*$.

Basic open sets are the sets $U_q = \{qn : n \in \mathbb{N}_+^\times\} \subseteq \mathbb{Q}_+^*$.
In particular, $U_1 = \mathbb{N}_+^\times$.

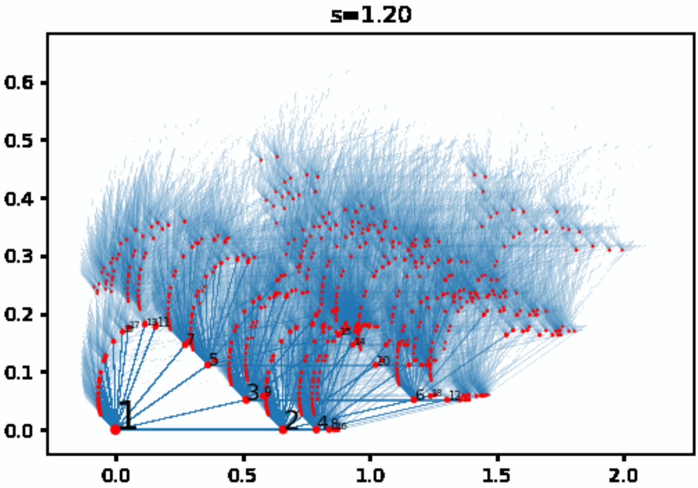
# Equivariant sheaves



# Equivariant sheaves



# Equivariant sheaves



# Equivariant sheaves

The elements of $\mathbb{A}^f/\widehat{\mathbb{Z}}^*$ are given by formal products $\prod_p p^{e_p}$ with $e_p \in \mathbb{Z} \cup \{+\infty\}$ for each prime $p$, and $e_p \geq 0$ for almost all primes.

This space is precisely the sobrification of $X = \mathbb{Q}_+^*$.

So $\mathbf{PSh}(\mathbb{N}_+^\times) \;\simeq\; \mathbf{Sh}_{\mathbb{Q}_+^*}(\mathbb{Q}_+^*) \;\simeq\; \mathbf{Sh}_{\mathbb{Q}_+^*}(\mathbb{A}^f/\widehat{\mathbb{Z}}^*)$.

Similarly, $\mathbf{PSh}(\mathrm{M}_n^{\mathrm{ns}}(\mathbb{Z})) \;\simeq\; \mathbf{Sh}_{\mathrm{GL}_n(\mathbb{Q})}(\mathrm{M}_n(\mathbb{A}^f)/\mathrm{GL}_n(\widehat{\mathbb{Z}}))$.

# Étale geometric morphisms

This part is based on joint work with Morgan Rogers.

An étale geometric morphism is a geometric morphism of the form

$$\mathcal{E}/X \xrightarrow{f} \mathcal{E}$$

with $f^*(E)$ given by $\pi_X : E \times X \to X$.

**Question:** what are the étale geometric morphisms

$$\mathbf{PSh}(M) \to \mathbf{PSh}(N),$$

with $M$ and $N$ monoids?

# Étale geometric morphisms

The points of $\mathcal{E}/X$ are given by pairs $(p, x)$ with $p$ a point of $\mathcal{E}$ and $x \in p^*X$.

The morphisms $(p, x) \to (q, y)$ are the morphisms $\eta : p^* \to q^*$ such that $\eta_X(x) = y$.

$$\mathbf{Pts}(\mathcal{E}/X) \simeq \int^{p \in \mathbf{Pts}(\mathcal{E})} p^*(X).$$

# Étale geometric morphisms

**Theorem (H, Rogers)**

*Let $\phi : M \to N$ be a monoid map. Then the induced geometric morphism $\mathbf{PSh}(M) \to \mathbf{PSh}(N)$ is étale if and only if*

1. *$\phi$ is injective;*
2. *if $a \in \phi(M)$ and $ab \in \phi(M)$ then $b \in \phi(M)$;*
3. *for every $n \in N$ there is some $u \in N^{\ltimes}$ such that $nu \in \phi(M)$.*

Example: the inclusion $\phi : \mathbb{N}_+^\times \hookrightarrow \mathbb{Z}^{\mathrm{ns}}$.

# Étale geometric morphisms

More precisely, $\mathbf{PSh}(\mathbb{N}_+^\times) \simeq \mathbf{PSh}(\mathbb{Z}^{\mathrm{ns}})/X$ with $X = \{-1, 1\}$ with the right $\mathbb{Z}^{\mathrm{ns}}$-action given by

$$x \cdot z \; = \; \begin{cases} x & \text{if } z > 0; \\ -x & \text{if } z < 0. \end{cases}$$

So the points of $\mathbf{PSh}(\mathbb{N}_+^\times)$ are pairs $(p, \sigma)$ where $p$ a point of $\mathbf{PSh}(\mathbb{Z}^{\mathrm{ns}})$ and $\sigma \in \{-1, 1\}$.

# Complete spreads

Let $\mathbf{PSh}(\mathcal{C})$ be a presheaf topos. Then a complete spread to $\mathbf{PSh}(\mathcal{C})$ is a geometric morphism of the form

$$\mathbf{PSh}\left(\int^{c \in C} F(c)\right) \to \mathbf{PSh}(\mathcal{C})$$

induced by a discrete opfibration

$$\int^{c \in C} F(c) \to \mathcal{C}$$

given by a functor $F : \mathcal{C} \to \mathbf{Sets}$.

# Complete spreads

For more on (complete) spreads, see the following works by Bunge–Funk:

- “Spreads and the symmetric topos”
- “Spreads and the symmetric topos II”
- “Singular Coverings of Toposes”

# Complete spreads

By the results of Bunge–Funk, the category of points of $\mathbf{PSh}(\int^N Y)$ is given by pairs $(A, \chi)$ where $A$ is a flat left $N$-set and $\chi : A \to Y$ a morphism of left $N$-sets.

# Complete spreads

**Theorem (H, Rogers)**

*Let $\phi : M \to N$ be a monoid map. Then the induced geometric morphism $\mathbf{PSh}(M) \to \mathbf{PSh}(N)$ is a complete spread if and only if*

1. *$\phi$ is injective;*
2. *if $b \in \phi(M)$ and $ab \in \phi(M)$ then $a \in \phi(M)$;*
3. *for every $n \in N$ there is some $v \in N^{\rtimes}$ such that $vn \in \phi(M)$.*

Example: the inclusion $\phi : \mathbb{N}_+^\times \hookrightarrow \mathbb{Z}^{\mathrm{ns}}$.

# Complete spreads

More precisely, $\mathbf{PSh}(\mathbb{N}_+^\times) \simeq \mathbf{PSh}(\int^{\mathbb{Z}^{\mathrm{ns}}} Y)$ where $Y = \{-1, 1\}$ with the left $\mathbb{Z}^{\mathrm{ns}}$-action given by

$$z \cdot x \;=\; \begin{cases} x & \text{if } z > 0; \\ -x & \text{if } z < 0. \end{cases}$$

So the points of $\mathbf{PSh}(\mathbb{N}_+^\times)$ are pairs $(A, \chi)$ with $A$ a left $\mathbb{Z}^{\mathrm{ns}}$-set and $\chi : A \to \{-1, 1\}$ a morphism of left $\mathbb{Z}^{\mathrm{ns}}$-sets.

Thank you!