Are precohesive geometric morphisms locally connected?

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Based on joint work with Morgan Rogers.

Our work started with a question by Thomas Streicher (which was based on discussions with Matías Menni).

Relevant articles

M. Menni and F. W. Lawvere, Internal choice holds in the discrete part of any cohesive topos satisfying stable connected codiscreteness, Theory and Applications of Categories (2015).

J. Hemelaer and M. Rogers, An essential, hyperconnected, local geometric morphism that is not locally connected, Applied Categorical Structures (2021), arXiv:2009.12241.

R. Garner and T. Streicher, An Essential Local Geometric Morphism which is not Locally Connected though its Inverse Image Part is an Exponential Ideal, Theory and Applications of Categories (2021), arXiv:2105.10143.

Essential, hyperconnected, local

Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. We say that:

- f is surjective if f^* is faithful;
- f is connected if f^* is fully faithful;
- f is hyperconnected if it is connected and f^* is closed under subquotients;
- f is local if it is connected and f_* has a right adjoint $f^!$;
- f is essential if f^* has a left adjoint $f_!$.

If f is essential and local, we have $f_! \dashv f^* \dashv f_* \dashv f^!$.

Precohesive geometric morphisms

A geometric morphism f is precohesive if it is essential, hyperconnected, local, with $f_!$ preserving finite products.

If f is essential, hyperconnected, local, then
$f_!$ preserves finite products $\Leftrightarrow$ f^* cartesian closed (Johnstone).

We say that f is locally connected if f^* is locally cartesian closed, i.e. if the slices

$$f/X : \mathcal{F}/f^*X \to \mathcal{E}/X$$

have cartesian closed inverse image functor, for every X in $\mathcal{E}$.

Stably precohesive

If f is precohesive and locally connected, then each slice

$$f/X : \mathcal{F}/f^*X \to \mathcal{E}/X$$

is again precohesive (Lawvere–Menni).

These morphisms are called stably precohesive, or punctually locally connected (Johnstone).

Questions

Is every precohesive geometric morphism locally connected?

We don't know!

More generally, is every essential, hyperconnected, local geometric morphism locally connected?

No. (H.–Rogers)

Alternatively, is every essential, local geometric morphism with $f_!$ preserving finite products locally connected?

No. (Garner–Streicher)

Presheaves on monoids

We consider the topos of presheaves $\mathbf{PSh}(M)$, where M is a monoid (a category with one object).

Every surjective monoid map $\phi : M \to N$ induces a hyperconnected essential geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$.

$$\begin{aligned} f_!(X) &\simeq X \otimes_M N \\ f^*(Y) &\simeq Y \\ f_*(X) &\simeq \mathcal{H}om_M(N, X). \end{aligned}$$

When is this geometric morphism local?

We already have connectedness, so f is local if and only if $f_* \simeq \mathcal{H}om_M(N, -)$ preserves colimits.

- $\mathcal{H}om_M(N, -)$ preserves coproducts
  $\Leftrightarrow$ N indecomposable/connected;
- $\mathcal{H}om_M(N, -)$ preserves filtered colimits
  $\Leftrightarrow$ N finitely presented;
- $\mathcal{H}om_M(N, -)$ preserves epimorphisms
  $\Leftrightarrow$ N projective;
- $\mathcal{H}om_M(N, -)$ preserves colimits
  $\Leftrightarrow$ N indecomposable projective.

Indecomposable projectives

The indecomposable projective right M-sets are of the form eM for $e \in M$ an idempotent.

So if we want $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ to be local,
we need to take $N = eM$.

Because $\phi : M \to N$ is a monoid map,
we need to take $N = eM = eMe$,
with ϕ given by $\phi(m) = eme$.

Essential + local + hyperconnected

Proposition

Let M be a monoid, and $e \in M$ an idempotent such that $eM = eMe$. Then $\phi : M \to eMe$, $\phi(m) = eme$ is a monoid map inducing an essential, hyperconnected, local geometric morphism

$$f : \mathbf{PSh}(M) \to \mathbf{PSh}(eMe).$$

Can we find an M and $e \in M$ such that f is not locally connected?

Situations where f is locally connected

Take a monoid M and an idempotent $e \in M$ such that $eM = eMe$.

If additionally $eMe = Me$, then eMe is indecomposable projective as left M-set as well. But then

$$f_! \simeq - \otimes_M eMe$$

preserves all colimits, in particular f is locally connected.

This happens whenever $eMe = Me$ (for example: M commutative, M cographical).

Counterexample

Take $M = \langle x, e : e^2 = e,\ xe = x \rangle$ with the idempotent $e \in M$.

Every element can be written as either x^n or ex^n for some $n \in \mathbb{N}$.
It follows that $eM = eMe$, and eMe is isomorphic to $(\mathbb{N}, +)$.

We claim that $f : \mathbf{PSh}(M) \to \mathbf{PSh}(eMe)$ is not locally connected.
Enough to show that f^* is not cartesian closed.

Counterexample

More precisely, $f^*(2^N) \longrightarrow f^*(2)^{f^*(N)}$ is not an isomorphism, for $2 = 1 \sqcup 1$.

The elements of $f^*(2^N)$ are:

$$\mathcal{H}om_M(M, f^*(2^N)) \;\simeq\; \mathcal{H}om_N(N, 2^N) \;\simeq\; \mathcal{H}om_N(N \times N, 2)$$

(i.e. the complemented subsets of $N \times N$ as right N-set).

The elements of $f^*(2)^{f^*(N)}$ are:

$$\mathcal{H}om_M(M, f^*(2)^{f^*(N)}) \;\simeq\; \mathcal{H}om_M(M \times f^*(N), f^*(2)).$$

(i.e. the complemented subsets of $M \times N$ as right M-set).

Counterexample

The comparison map sends a complemented subobject $S \subseteq N \times N$ to the complemented subobject

$$\{(m, n) \in M \times N : (\phi(m), n) \in S\}.$$

Now show that $T = \{(ex^{n+1}, ex^n) : n \geq 0\}$ is a complemented subobject of $M \times N$ that does not come from a complemented subobject of $N \times N$.

More topos theory!

Previous proof is a bit technical, and difficult to generalize to other monoids M.

In the following, we present an alternative proof that

$$f : \mathbf{PSh}(M) \to \mathbf{PSh}(eMe)$$

is not locally connected.

The alternative proof is more geometrical: we use more topos theory and less algebra.

Minimal toposes, minimal monoids

A topos is Boolean if it does not have a dense subtopos (other than the topos itself).

We call a topos minimal if it does not have a pure subtopos (other than the topos itself).

A pure subtopos is a subtopos containing $1 \sqcup 1$.
Pure subtoposes are dense, so Boolean toposes are minimal.

We say a monoid M is minimal if $\mathbf{PSh}(M)$ is minimal.

Minimal toposes, minimal monoids

Examples of minimal toposes:

- $\mathbf{Sh}(\mathbb{R})$ (Bunge–Funk), so also $\mathbf{Sh}(S^1)$;
- $\mathbf{Sh}(X)$ with X a discrete space;
- $\mathbf{PSh}(M)$ for M a group or M a free monoid on ≥ 2 generators;
- $\mathbf{PSh}(M)$ with $M - M^{\ltimes}$ not connected as right M-set.

Examples of toposes that are not minimal:

- $\mathbf{Sh}(\mathbb{R}^2)$ (Bunge–Funk), so also $\mathbf{Sh}(M)$ for M a manifold of dimension ≥ 2;
- $\mathbf{PSh}(M)$ with M commutative and not a group;
- $\mathbf{PSh}(M)$ where M has a right calculus of fractions and is not a group.

Why is this relevant?

Proposition

Let $f : \mathcal{F} \to \mathcal{E}$ be a surjective geometric morphism, with $\mathcal{F}$ minimal and $\mathcal{E}$ not minimal. Then f is not locally connected.

Proof.

If f is locally connected, then the pullback of a pure subtopos $\mathcal{E}' \subseteq \mathcal{E}$ along f gives a pure subtopos $\mathcal{F}' \subseteq \mathcal{F}$ (Bunge–Funk). Suppose that $\mathcal{E}' \neq \mathcal{E}$. Then also $\mathcal{F}' \neq \mathcal{F}$, because f is surjective. But this is impossible, because $\mathcal{F}$ is minimal. $\square$

Back to the counterexample

$$M = \langle x, e : e^2 = e, xe = x \rangle.$$

Then $M^\ltimes = \{1\}$. We can show that $M - \{1\}$ is disconnected as right M-set.

So $\mathbf{PSh}(M)$ is minimal, and $\mathbf{PSh}(N) \simeq \mathbf{PSh}(\mathbb{N})$ is not. So there can be no locally connected surjection

$$f : \mathbf{PSh}(M) \to \mathbf{PSh}(N).$$

More generally

More generally, we can start with any N that is not minimal, $N^{\ltimes} = N^{\rtimes} = \{1\}$.

Let M be the monoid with as elements n and en for $n \in N$. Multiplication extends multiplication on N, with $e^2 = e$ and $ne = n$ for every $n \in N$, $n \neq 1$.

Again, $N = eM = eMe$, so we get an essential, hyperconnected, local geometric morphism

$$f : \mathbf{PSh}(M) \to \mathbf{PSh}(N).$$

Further, $M - \{1\}$ is disconnected, so M is minimal, so f is not locally connected.

The problem remains open

Unfortunately, in this more general construction, we still have that f^* is not cartesian closed.

So the original question remains open: it is not known whether precohesive geometric morphisms are locally connected.

Thank you!