GEOMETRIC MORPHISMS TO SLICE TOPOSES

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We give a proof for a result from SGA4 [sga72] that describes the geometric morphisms $g : \mathcal{F} \to \mathcal{E}/X$ in terms of pairs (f, a) where $f : \mathcal{F} \to \mathcal{E}$ is a geometric morphism and $a : 1 \to f^*(X)$ is a global element. This result appears as Proposition 5.12 in SGA4, Exposé IV, and the proof given here will be very similar to the proof in SGA4.

Throughout, we fix a Grothendieck topos $\mathcal{E}$ with an object X in $\mathcal{E}$, and we will write

$$x : \mathcal{E}/X \to \mathcal{E}$$

for the projection geometric morphism associated to the slice topos $\mathcal{E}/X$. The goal is to determine the category of geometric morphisms $\mathbf{Geom}(\mathcal{F}, \mathcal{E}/X)$ for any Grothendieck topos $\mathcal{F}$, in terms of $\mathbf{Geom}(\mathcal{F}, \mathcal{E})$. We use the standard convention that a map of geometric morphisms $g \to g'$ is given by a natural transformation $g^* \Rightarrow (g')^*$. Note that SGA4 uses the opposite convention. We will show that the functor

$$\mathbf{Geom}(\mathcal{F}, \mathcal{E}/X) \longrightarrow \mathbf{Geom}(\mathcal{F}, \mathcal{E})$$

induced by x is a discrete opfibration, corresponding to the copresheaf that sends f to $\mathrm{Hom}_{\mathcal{F}}(1, f^*(X))$. In symbols:

$$\mathbf{Geom}(\mathcal{F}, \mathcal{E}/X) \simeq \int^{f \in \mathbf{Geom}(\mathcal{F}, \mathcal{E})} \mathrm{Hom}_{\mathcal{F}}(1, f^*(X)). \tag{1}$$

Spelled out, this means that $\mathbf{Geom}(\mathcal{F}, \mathcal{E}/X)$ is equivalent to the category with

- as objects the pairs (f, a) with $f : \mathcal{F} \to \mathcal{E}$ a geometric morphism and $a : 1 \to f^*(X)$ a map in $\mathcal{F}$;
- as morphisms $(f, a) \to (f', a')$ the natural transformations $\eta : f^* \Rightarrow (f')^*$ such that the diagram

$$\begin{array}{ccc} 1 & \xrightarrow{\ a\ } & f^*(X) \\ & \searrow^{a'} & \downarrow{\scriptstyle \eta_X} \\ & & (f')^*(X) \end{array}$$

  commutes.

For a fixed geometric morphism $f : \mathcal{F} \to \mathcal{E}$ we denote by $\mathbf{Geom}_{\mathcal{E}}(\mathcal{F}, \mathcal{E}/X)$ the category of morphisms $\mathcal{F} \to \mathcal{E}/X$ relative over $\mathcal{E}$. The category $\mathbf{Geom}_{\mathcal{E}}(\mathcal{F}, \mathcal{E}/X)$ is precisely the fiber of the discrete opfibration $\mathbf{Geom}(\mathcal{F}, \mathcal{E}/X) \longrightarrow \mathbf{Geom}(\mathcal{F}, \mathcal{E})$ above f. It has as objects the morphisms $g : \mathcal{F} \to \mathcal{E}/X$ such that $xg \simeq f$, and as morphisms $g \to g'$ the natural transformations $\eta : g^* \Rightarrow (g')^*$ such that $x\eta : f^* \Rightarrow f^*$ is the trivial natural transformation. From (1) it follows that

$$\mathbf{Geom}_{\mathcal{E}}(\mathcal{F}, \mathcal{E}/X) \simeq \mathrm{Hom}_{\mathcal{F}}(1, f^*(X)), \tag{2}$$

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by which we mean that $\mathbf{Geom}_{\mathcal{E}}(\mathcal{F}, \mathcal{E}/X)$ is the discrete category corresponding to the set $\mathrm{Hom}_{\mathcal{F}}(1, f^*(X))$. In the special case where $\mathcal{F} \simeq \mathcal{E}/Y$ for some object Y in $\mathcal{E}$, and f is the projection geometric morphism $y : \mathcal{E}/Y \to \mathcal{E}$, we see that

$$\mathbf{Geom}_{\mathcal{E}}(\mathcal{E}/Y, \mathcal{E}/X) \;\simeq\; \mathrm{Hom}_{\mathcal{E}/Y}(1, y^*(X)) \;\simeq\; \mathrm{Hom}_{\mathcal{E}}(y_!(1), X) \;\simeq\; \mathrm{Hom}_{\mathcal{E}}(Y, X).$$

This result appears as Exercise 5.14 in SGA4, Exposé IV (as an application of (1)). A similar, direct proof is given in the recent paper by Caramello and Zanfa [CZ21, Lemma 6.1.2], where the result is put in a wider context.

Finding references or elegant proofs for $\mathbf{Geom}_{\mathcal{E}}(\mathcal{E}/Y, \mathcal{E}/X) \simeq \mathrm{Hom}_{\mathcal{E}}(Y, X)$ was the original motivation to look at the result discussed in this note. I thank Morgan Rogers and Steve Vickers for interesting discussions on this topic. I expect that there are other interesting proofs of $\mathbf{Geom}_{\mathcal{E}}(\mathcal{E}/Y, \mathcal{E}/X) \simeq \mathrm{Hom}_{\mathcal{E}}(Y, X)$; if you know any, please let me know.

1. The proof

For an object $\varphi : E \to X$ in $\mathcal{E}/X$, we consider the diagram

$$\begin{array}{ccccc} E & \xrightarrow{1\times\varphi} & E\times X & \overset{\pi_1\times\pi_2\times\pi_2}{\underset{\pi_1\times\varphi\pi_1\times\pi_2}{\rightrightarrows}} & E\times X\times X \\ & {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\pi_2} & \swarrow{\scriptstyle\pi_3} & \\ & & X & & \end{array} \tag{3}$$

in $\mathcal{E}/X$. Our notations are as follows: we write π_i for the projection on the ith component, and for maps $f_i : Z \to Z_i$ with $i \in \{1, 2, \ldots, n\}$ we write $f_1 \times \cdots \times f_n$ for the corresponding map $Z \to Z_1 \times \cdots \times Z_n$.

Lemma 1. *Diagram* (3) *above is an equalizer.*

Proof. Let $\psi : Z \to E \times X$ be a morphism in $\mathcal{E}/X$ such that $(\pi_1 \times \pi_2 \times \pi_2)\psi = (\pi_1 \times \varphi\pi_1 \times \pi_2)\psi$. This is the case precisely when $\pi_2\psi = \varphi\pi_1\psi$. This means that the map $\pi_1\psi : Z \to E$ satisfies $(1 \times \varphi)\pi_1\psi = \psi \times \varphi\psi$, so a lift $Z \to E$ exists. If $h : Z \to E$ is another map that makes the diagram

$$\begin{array}{ccc} Z & & \\ {\scriptstyle h}\downarrow & \searrow{\scriptstyle\psi} & \\ E & \xrightarrow[1\times\varphi]{} & E\times X \end{array}$$

commute, then we see that $\pi_1\psi = \pi_1(1 \times \varphi)h = h$, so $h = \pi_1\psi$ is the unique lift. $\square$

Note that in this way we have written $\varphi : E \to X$ as an equalizer of two objects in the image of x^*. Further, note that the map $\pi_1 \times \varphi\pi_1 \times \pi_2$ is the image of $1 \times \varphi$ along x^*, while $\pi_1 \times \pi_2 \times \pi_2$ does not lie in the image of x^*. In more compact notation, we can say that φ is the equalizer of the diagram

$$x^*(E) \overset{\pi_1\times\Delta\pi}{\underset{x^*(1\times\varphi)}{\rightrightarrows}} x^*(E) \times x^*(X) \tag{4}$$

where $\times$ now denotes the product in $\mathcal{E}/X$, π denotes the map to the terminal object, and $\Delta : 1 \to x^*(X)$ is the map in $\mathcal{E}/X$ corresponding to the diagonal map $X \to X \times X$ (the terminal object 1 is given by the identity $X \to X$ and $x^*(X)$ is given by $\pi_2 : X \times X \to X$).

Proposition 2. *Let $g : \mathcal{F} \to \mathcal{E}/X$ be a geometric morphism and let $f : \mathcal{F} \to \mathcal{E}$ be the composition $f = xg$. Then g^* is completely determined by f^* and by the image of the map $\Delta : 1 \to x^*(X)$ in $\mathcal{E}/X$, as defined above.*

Proof. For an object $\varphi : E \to X$ in $\mathcal{E}/X$, we can write φ as an equalizer as in diagram (4). Because $xg = f$ and g preserves finite products and equalizers, we then see that $g^*(\varphi)$ is the equalizer of

$$f^*(E) \underset{f^*(1\times\varphi)}{\overset{1\times g^*(\Delta)\pi}{\rightrightarrows}} f^*(E) \times f^*(X)$$

which as promised only depends on f and $g^*(\Delta)$. Further, if there is another object $\varphi' : E' \to X$ and a morphism $\psi : E \to E'$ in $\mathcal{E}/X$ (which means that $\varphi'\psi = \varphi$), then ψ arises from taking equalizers for the diagram

$$\begin{array}{ccc} x^*(E) & \underset{x^*(1\times\varphi)}{\overset{\pi_1\times\Delta\pi}{\rightrightarrows}} & x^*(E) \times x^*(X) \\ {\scriptstyle x^*(\psi)}\downarrow & & \downarrow{\scriptstyle x^*(\psi\pi_1\times\pi_2)} \\ x^*(E') & \underset{x^*(1\times\varphi')}{\overset{\pi_1\times\Delta\pi}{\rightrightarrows}} & x^*(E') \times x^*(X). \end{array}$$

So $g^*(\psi)$ is the map arising from taking equalizers for the diagram

$$\begin{array}{ccc} f^*(E) & \underset{f^*(1\times\varphi)}{\overset{1\times g^*(\Delta)\pi}{\rightrightarrows}} & f^*(E) \times f^*(X) \\ {\scriptstyle f^*(\psi)}\downarrow & & \downarrow{\scriptstyle f^*(\psi\pi_1\times\pi_2)} \\ f^*(E') & \underset{f^*(1\times\varphi')}{\overset{1\times g^*(\Delta)\pi}{\rightrightarrows}} & f^*(E') \times f^*(X). \end{array}$$

This shows that also $g^*(\psi)$ depends only on f and $g^*(\Delta)$. □

We conclude that the geometric morphism $g : \mathcal{F} \to \mathcal{E}/X$ is completely determined by the map $f : \mathcal{F} \to \mathcal{E}$ given by $f = xg$, together with a choice of map $a : 1 \to f^*(X)$ (namely $a = g^*(\Delta)$). It turns out that a converse holds as well:

Proposition 3. *Fix a geometric morphism $f : \mathcal{F} \to \mathcal{E}$. Every choice of map $a : 1 \to f^*(X)$ is of the form $g^*(\Delta)$ for some geometric morphism $g : \mathcal{F} \to \mathcal{E}$ with $xg \simeq f$.*

Proof. The geometric morphism $f : \mathcal{F} \to \mathcal{E}$ induces a geometric morphism $(f/X) : \mathcal{F}/f^*(X) \to \mathcal{E}/X$ that sends $\varphi : E \to X$ to $f^*(\varphi) : f^*(E) \to f^*(X)$. The functor $(f/X)^*$ sends the map $\Delta : 1 \to x^*(X)$ to the map in $\mathcal{F}/f^*(X)$ corresponding to the diagonal map

$$\begin{array}{ccc} f^*(X) & \xrightarrow{1\times 1} & f^*(X) \times f^*(X) \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \pi_2} \\ & & f^*(X). \end{array} \tag{5}$$

Further, there is a geometric morphism $\mathcal{F}/a : \mathcal{F} \to \mathcal{F}/f^*(X)$ corresponding to $a : 1 \to f^*(X)$. Its inverse image functor $(\mathcal{F}/a)^*$ sends a map $\sigma : Y \to f^*(X)$ to the pullback along a, as follows:

$$\begin{array}{ccc} (\mathcal{F}/a)^*(\sigma) & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle \sigma} \\ 1 & \xrightarrow{a} & f^*(X). \end{array}$$

On morphims, $(\mathcal{F}/a)^*$ is similarly defined via the pullback construction. We now see that the pullback of the diagonal map of (5) (along a) is precisely given by

$$a : 1 \to f^*(X).$$

It follows that $(\mathcal{F}/a)^*(f/X)^*(\Delta) = a$, and we conclude that the geometric morphism $g = (f/X) \circ (\mathcal{F}/a)$ satisfies the requirements of the proposition. □

This gives a correspondence between geometric morphisms $g : \mathcal{F} \to \mathcal{E}/X$ and pairs (f, a), where f is a geometric morphism $\mathcal{F} \to \mathcal{E}$ and $a : 1 \to f^*(X)$ is a map in $\mathcal{F}$. It remains to determine the maps (geometric transformations) $g \to g'$ for two geometric morphisms $g, g' : \mathcal{F} \to \mathcal{E}/X$.

Proposition 4. *Let $g, g' : \mathcal{F} \to \mathcal{E}/X$ be two geometric morphisms, with g corresponding to the pair (f, a) and g' corresponding to the pair (f', a'), where $f, f' : \mathcal{F} \to \mathcal{E}$ are geometric morphisms and $a : 1 \to f^*(X)$, $a' : 1 \to (f')^*(X)$ maps in $\mathcal{F}$.*

Fix a natural transformation $\eta : f^* \Rightarrow (f')^*$. Then there is at most one natural transformation $\tilde{\eta} : g^* \Rightarrow (g')^*$ such that $\tilde{\eta}x^* = \eta$. Moreover, such a $\tilde{\eta}$ exists if and only if the diagram

$$\begin{array}{ccc} 1 & \xrightarrow{a} & f^*(X) \\ & {\scriptstyle a'}\searrow & \downarrow{\scriptstyle \eta_X} \\ & & (f')^*(X) \end{array}$$

commutes.

Proof. Let $\tilde{\eta} : g^* \Rightarrow (g')^*$ be a natural transformation such that $\tilde{\eta}x^* = \eta$. Let $\varphi : E \to X$ be an object in $\mathcal{E}/X$. We know that $g^*(\varphi)$ and $(g')^*(\varphi)$ can be computed as the equalizers of respectively

$$f^*(E) \underset{f^*(1\times\varphi)}{\overset{1\times a\pi}{\rightrightarrows}} f^*(E) \times f^*(X)$$

and

$$(f')^*(E) \underset{(f')^*(1\times\varphi)}{\overset{1\times a'\pi}{\rightrightarrows}} (f')^*(E) \times (f')^*(X).$$

In particular, there is a commutative diagram

$$\begin{array}{ccc} g^*(\varphi) & \longrightarrow & f^*(E) \\ {\scriptstyle \tilde{\eta}_\varphi}\downarrow & & \downarrow{\scriptstyle \tilde{\eta}_{x^*(E)}} \\ (g')^*(\varphi) & \longrightarrow & (f')^*(E) \end{array}$$

with the horizontal maps monomorphisms. By using that the lower horizontal map is a monomorphism, we see that $\tilde{\eta}_\varphi$ is completely determined by $\tilde{\eta}_{x^*(E)}$. Moreover, because $\tilde{\eta}x^* = \eta$, we see that $\tilde{\eta}_{x^*(E)} = \eta_E$, and as a result it only depends on η. So $\tilde{\eta}$ is uniquely determined by η.

Further, if a natural transformation $\tilde{\eta}$ exists such that $\tilde{\eta}x^* = \eta$, then we also have a commuting diagram

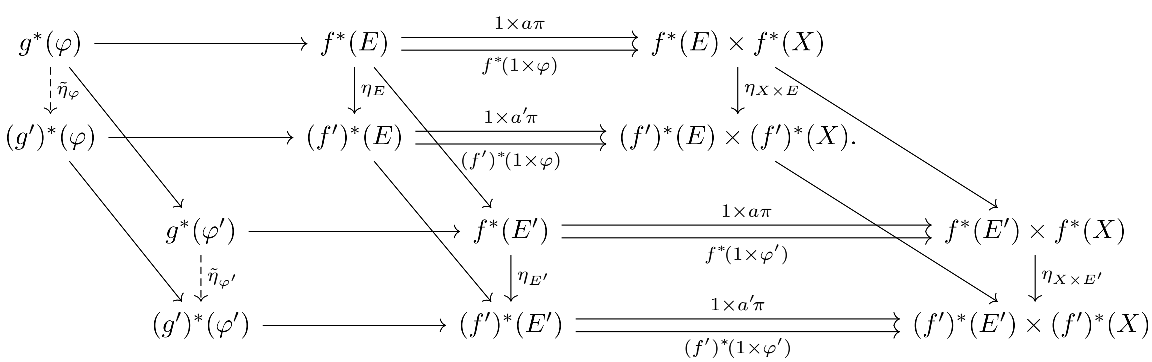
$$\begin{array}{ccc} f^*(E) & \xrightarrow{1\times a\pi} & f^*(E)\times f^*(X) \\ {\scriptstyle \tilde{\eta}_{x^*(E)}}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\eta}_{x^*(E\times X)}} \\ (f')^*(E) & \xrightarrow[1\times a'\pi]{} & (f')^*(E)\times (f')^*(X). \end{array}$$

with again $\tilde{\eta}_{x^*(E)} = \eta_E$ and further $\tilde{\eta}_{x^*(E\times X)} = \eta_{E\times X} = \eta_E\pi_1 \times \eta_X\pi_2$. Recall that π was our notation for the map to the terminal object. By considering the special case where E is the terminal object in $\mathcal{E}$, we get $\eta_X a = a'$, which shows the commutativity of the diagram in the statement.

It remains to show the converse statement that if $\eta_X a = a'$, then there exists a natural transformation $\tilde{\eta}$ such that $\tilde{\eta}x^* = \eta$. Let $\varphi : E \to X$ be an object in $\mathcal{E}/X$. We construct $\tilde{\eta}_\varphi$, by via taking equalizers as follows:

$$\begin{array}{ccccc} g^*(\varphi) & \longrightarrow & f^*(E) & \underset{f^*(1\times\varphi)}{\overset{1\times a\pi}{\rightrightarrows}} & f^*(E)\times f^*(X) \\ {\scriptstyle \tilde{\eta}_\varphi}\,\vdots\!\downarrow & & \big\downarrow{\scriptstyle \eta_E} & & \big\downarrow{\scriptstyle \eta_{X\times E}} \\ (g')^*(\varphi) & \longrightarrow & (f')^*(E) & \underset{(f')^*(1\times\varphi)}{\overset{1\times a'\pi}{\rightrightarrows}} & (f')^*(E)\times (f')^*(X). \end{array}$$

To show that $\tilde{\eta}$ defines a natural transformation, consider the diagram



for a morphism $\psi : \varphi \to \varphi'$ in $\mathcal{E}/X$, with diagonal morphisms given from left to right by $g^*(\psi)$, $f^*(\psi)$, $f^*(\psi\pi_1\times\pi_2)$ for the upper row and by $(g')^*(\psi)$, $(f')^*(\psi)$, $(f')^*(\psi\pi_1\times\pi_2)$ for the lower row.

To show that $\tilde{\eta}$ is a natural transformation, or in other words that $\tilde{\eta}_{\varphi'}g^*(\psi) = (g')^*(\psi)\tilde{\eta}_\varphi$, it is enough to show that the following six equations hold:

$$f^*(\psi\pi_1\times\pi_2)(1\times a\pi) = (1\times a\pi)f^*(\psi)$$
$$(f')^*(\psi\pi_1\times\pi_2)(1\times a'\pi) = (1\times a'\pi)(f')^*(\psi)$$
$$f^*(\psi\pi_1\times\pi_2)f^*(1\times\varphi) = f^*(1\times\varphi')f^*(\psi)$$
$$(f')^*(\psi\pi_1\times\pi_2)(f')^*(1\times\varphi) = (f')^*(1\times\varphi')(f')^*(\psi)$$
$$\nu_{E'}f^*(\psi) = (f')^*(\psi)\nu_E$$
$$\nu_{E'\times X}f^*(\psi\pi_1\times\pi_2) = (f')^*(\psi\pi_1\times\pi_2)\nu_{E\times X}$$

corresponding to six commutative squares in the diagram. For the first equation, we use that $f^*(\psi\pi_1\times\pi_2) = f^*(\psi)\pi_1\times\pi_2$ (which holds because f^* preserves products), and we compute

$$f^*(\psi\pi_1 \times \pi_2)(1 \times a\pi) = (f^*(\psi)\pi_1 \times \pi_2)(1 \times a\pi) = f^*(\psi) \times a\pi = (1 \times a\pi)f^*(\psi).$$

The second equation is analogous. The third and the fourth equation follow from

$$(\psi\pi_1 \times \pi_2)(1 \times \varphi) = \psi \times \varphi = \psi \times \varphi'\psi = (1 \times \varphi')\psi.$$

by applying f^* resp. $(f')^*$. The last two equations follow by naturality of η. This shows that $\tilde{\eta}$ is a natural transformation. □

References

[CZ21] Olivia Caramello and Riccardo Zanfa, **Relative topos theory via stacks**, preprint (2021), arXiv:2107.04417v1.

[sga72] **Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos**, Lecture Notes in Mathematics, Vol. 269, Springer-Verlag, Berlin-New York, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

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