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Topos-theoretic methods in noncommutative geometry

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Contents

Introduction

In this thesis, we will use topos theory to study the geometry behind some algebraic structures, such as:

- the nonzero natural numbers under multiplication (Chapter 3);
- the 2×2 integer matrices with nonzero determinant, under multiplication (Chapter 4);
- the category of Azumaya algebras and center-preserving algebra morphisms between them (Chapter 5).

In the introduction, we give a preview of some of the ideas in the thesis, focusing on the second example of 2×2 integer matrices. But first:

What is a topos?

The first step leading to the notion of topos is replacing a topological space X by the category of sheaves on X.

Recall that a *presheaf* $\mathcal{F}$ on X consists of a set $\mathcal{F}(U)$ for every open subset $U \subseteq X$, together with maps

$$\rho_{VU} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V) \tag{1}$$

whenever $V \subseteq U$, such that $\rho_{WV} \circ \rho_{VU} = \rho_{WU}$. Geometrically, we can interpret $s \in \mathcal{F}(U)$ as some kind of function defined on U. The maps ρ_{VU} then correspond to restriction, which is why we will use the shorthand

$$s|_V = \rho_{VU}(s). \tag{2}$$

We say that $\mathcal{F}$ is a *sheaf* if for any open cover $U = \bigcup_{i\in I} U_i$ we have an equality

$$\mathcal{F}(U) = \{(s_i)_{i\in I} : s_i \in \mathcal{F}(U_i),\ s_i|_{U_i\cap U_j} = s_j|_{U_i \cap U_j}\}. \tag{3}$$

The category $\mathsf{Sh}(X)$ of sheaves on X has properties similar to the properties of the category of sets. Moreover, we can reconstruct X from $\mathsf{Sh}(X)$ if X is Hausdorff (or more generally, if X is sober).

The definition of a sheaf on X only depends on the poset $\mathcal{O}(X)$ of open subsets $U \subseteq X$. We can interpret $\mathcal{O}(X)$ as a category, with the open sets as objects and a morphism $V \to U$ whenever $V \subseteq U$. A presheaf on X is then the same as a functor

$$\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \longrightarrow \mathsf{Sets} \tag{4}$$

to the category of sets, and the sheaf property is the same as above.

Expressing sheaves as some kind of functor opens the door to several generalizations. The poset $\mathcal{O}(X)$ is a *frame*: you can take arbitrary unions and finite intersections, and these satisfy

$$V \cap \left(\bigcup_{i \in I} U_i \right) = \bigcup_{i \in I} (V \cap U_i). \tag{5}$$

It turns out that not every frame can be written as $\mathcal{O}(X)$ for some topological space X, but this does not stop us from defining sheaves with respect to arbitrary frames. Again, the category of sheaves on a frame has many good properties, and we can completely reconstruct the frame from the category of sheaves on it.

The advantage of this approach is that we can now start thinking about frames geometrically: an arbitrary frame is interpreted as the poset of open subsets of a new notion of space that is (by definition) called a *locale*. We can define a point of a topological space in terms of a category of sheaves, and this can in turn be used to define points for a locale. The difference between locales and topological spaces is that two open sets in a locale are not necessarily equal if they contain the same points. Some nontrivial locales even have no points at all, which is why the study of locales is sometimes called pointless topology. While topological spaces are related to classical logic, locales can be used to model constructive logic.

In order to get to (Grothendieck) toposes, we need to replace the poset $\mathcal{O}(X)$ by an arbitrary small category $\mathcal{C}$. A presheaf is then defined as a (covariant) functor

$$\mathcal{F} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Sets} \tag{6}$$

to the category of sets. So for each object C of $\mathcal{C}$ there is a set of sections $\mathcal{F}(C)$, and for all morphisms $f : D \longrightarrow C$ there are restriction maps

$$\rho_f : \mathcal{F}(C) \to \mathcal{F}(D) \tag{7}$$

such that $\rho_{f \circ g} = \rho_g \circ \rho_f$. In order to define sheaves, we first have to specify when an object C is *covered* by a family of morphisms

$$f_i : C_i \longrightarrow C,\ i \in I. \tag{8}$$

For some categories, there might be an obvious way to define coverings (for example for the category of open sets of a topological space, as above). In general though, there are many possibilities, each leading to a different definition of sheaves. This is formalized using *Grothendieck topologies*. A presheaf $\mathcal{F}$ is then a sheaf with respect to a certain Grothendieck topology J if

$$\mathcal{F}(C) = \{(s_i)_{i \in I} : s_i \in \mathcal{F}(C_i),\ \rho_g(s_i) = \rho_h(s_j) \text{ whenever } f_i \circ g = f_j \circ h\} \tag{9}$$

for each J-covering consisting of morphisms $f_i : C_i \to C,\ i \in I$. The category of sheaves is denoted by $\mathsf{Sh}(\mathcal{C}, J)$. Again, the category of sheaves has many good properties, and if we know $\mathcal{C}$ and $\mathsf{Sh}(\mathcal{C}, J)$ then we can reconstruct J.

We say that a category is a *(Grothendieck) topos* if it is equivalent to the category of sheaves on a small category $\mathcal{C}$ equipped with a Grothendieck topology J. Note that two couples $(\mathcal{C}, J)$ and $(\mathcal{C}', J')$ can have equivalent categories of sheaves $\mathsf{Sh}(\mathcal{C}, J) \simeq \mathsf{Sh}(\mathcal{C}', J')$. For example, suppose that $\mathcal{C}$ is equivalent to a poset, i.e. there is at most one morphism between two objects. Then $\mathsf{Sh}(\mathcal{C}, J)$ is always equivalent to the category of sheaves on some locale. For more on this situation, we refer to Chapter 2, where we will try to classify all Grothendieck topologies on a poset, as explicitly as possible.

The concrete examples of toposes that we study in this thesis, always have *enough points* (for a definition, see Chapter 1). They fail to be topological spaces in a different way than how some locales fail to be topological spaces. For geometric intuition about these toposes, we should mention the result by Butz–Moerdijk [BM98] that every topos with enough points is equivalent to the category of sheaves on some topological groupoid. So we can imagine these toposes as topological spaces with the additional data of continuous group actions on the points.

The topos of $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-sets

As an example of a topos that is not equivalent to the category of sheaves on a locale, we mention the following topos that is studied in Chapter 4. Consider the category $\mathcal{C}$ with one object $*$ and a morphism $* \xrightarrow{a} *$ for every $a \in \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$, with

$$\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}) = \{a \in \mathrm{M}_2(\mathbb{Z}) : \det(a) \neq 0\} \tag{10}$$

the 2×2 integer matrices with nonzero determinant. Composition of morphisms is defined as $a \circ b = ba$. We take the trivial Grothendieck topology, so all presheaves on $\mathcal{C}$ are sheaves. The resulting topos can alternatively be described as the category $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$ of sets with a left action of the monoid $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$.

How can we describe a topos like this geometrically? First of all, we can compute the topos-theoretic points. For $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$, we will show in Chapter 4 that the points (up to isomorphism) are classified by the double quotient

$$\mathrm{GL}_2(\widehat{\mathbb{Z}}) \backslash \mathrm{M}_2(\mathbb{A}_f) / \mathrm{GL}_2(\mathbb{Q}). \tag{11}$$

Here $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ is the ring of profinite integers (the product is over all prime numbers p). Further, $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$ is the ring of finite adeles. The double quotient (11) also classifies the abelian groups M with $\mathbb{Z}^2 \subseteq M \subseteq \mathbb{Q}^2$ up to isomorphism, and moreover it is related to the double coset space in the Langlands program. There is no known concrete classification of the abelian groups $\mathbb{Z}^2 \subseteq M \subseteq \mathbb{Q}^2$ up to isomorphism, which suggests that the double quotient is very complicated. The advantages of topos theory in this case are that:

(a) we can find a geometrical interpretation for the double quotient (11), in terms of a topos;
(b) this topos is completely determined by multiplication in $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$.

While we can equivalently describe the double quotient using a topological space equipped with a group action, the relation with $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ would be less clear. With the topos-theoretical approach we have both sides of the picture: the algebraic side (factorization of integer matrices) and the geometric side (a space with a group action).

We can study the topos $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ combinatorially by looking at the poset

$$\mathrm{GL}_2(\mathbb{Z}) \backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}).$$

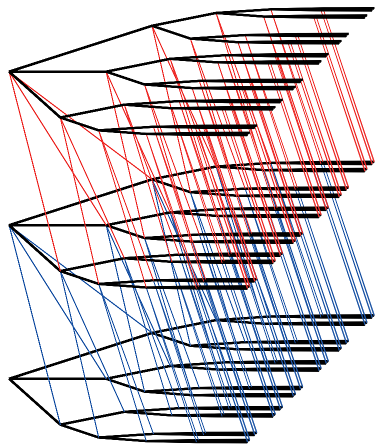


Figure 1: A (truncated) Hasse diagram of $\mathrm{GL}_2(\mathbb{Z}_2)\backslash\,\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}_2)$.

The elements are equivalence classes of matrices in $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$, with $a \sim b$ if $b = ua$ for some $u \in \mathrm{GL}_2(\mathbb{Z})$. The partial order is then defined as $b \geq a$ whenever $b = ma$ for some $m \in \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$. The resulting poset is a (restricted) product of similarly defined posets $\mathrm{GL}_2(\mathbb{Z}_p)\backslash\,\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}_p)$ for each prime p. The Hasse diagram in the case $p = 2$ is shown above (and on the cover).

Elements are represented by vertices, and $a \leq b$ if we can find a path from a to b by successively going to the right or upwards. The poset $\mathrm{GL}_2(\mathbb{Z}_2)\backslash\,\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}_2)$ consists of different levels (of which the lowest three are drawn). Each level is, as a graph, isomorphic to the Bruhat-Tits tree for the group $\mathrm{SL}_2(\mathbb{Q}_2)$. The situation for the primes $p \neq 2$ is analogous. If we combine the Bruhat-Tits trees for all primes p, then we get a graph isomorphic to Conway's big picture, which is related to congruence subgroups and monstrous moonshine, see Le Bruyn [LB18].

The study of the topos $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$ was originally motivated by the work of Connes and Consani on their Arithmetic Site. The topos associated to the Arithmetic Site is the category $\mathbb{N}_+^{\times}$-Sets of sets with an action of $\mathbb{N}_+^{\times}$, where $\mathbb{N}_+^{\times}$ denotes the nonzero natural numbers under multiplication. Connes and Consani use this topos, in combination with a sheaf of idempotent semirings, in their approach to the Riemann Hypothesis. In Chapter 3, we will study the topos $\mathbb{N}_+^{\times}$-Sets in detail.

Overview of the thesis

In Chapter 1, we will give a short introduction to topos theory, from a geometrical point of view. We also fix some notations that we will use in the other chapters.

In Chapter 2, we study Grothendieck topologies on general posets. In particular, we give an explicit description of the Grothendieck topologies with enough points and the Grothendieck topologies of finite type. As an application, we compute the cardinalities of these families of Grothendieck topologies for some example posets.

In Chapter 3, we analyze the topos $\mathbb{N}_+^{\times}$-Sets, which is part of the Connes–Consani Arithmetic Site. The points of this topos were already known, but we will discuss two alternative ways to compute them, that are easily generalized to some related toposes. Afterwards, we apply the results of the previous chapter in order to study Grothendieck topologies on Conway's big cell (the poset of nonzero natural numbers with $m \leq n \Leftrightarrow n \mid m$). These will in turn describe the subtoposes of $\mathbb{N}_+^\times$-Sets.

In Chapter 4, we study the topos $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$ discussed above. We show that the topos-theoretic points are classified by the double quotient

$$\mathrm{GL}_2(\widehat{\mathbb{Z}}) \backslash \mathrm{M}_2(\mathbb{A}_f) / \mathrm{GL}_2(\mathbb{Q})$$

and we explain the relation to torsion-free abelian groups of rank 2. Then we discuss the combinatorial description using the poset $\mathrm{GL}_2(\mathbb{Z}) \backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ and show how Conway's big picture appears as a subgraph of its Hasse diagram. Next, we compute the automorphisms of $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$ and we will see that they act trivially on the points of the topos. Afterwards, we show that the similar results hold if we replace $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ by the $ax+b$-monoid. We end the chapter by discussing the relationship between the double quotient and a conjecture by Goormaghtigh regarding the integer solutions (x, y, m, n) to the equation

$$\frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1}.$$

In Chapter 5, we will consider topos-theoretical methods in a different situation, namely the study of Azumaya algebras. We first construct families of Grothendieck topologies on the opposite of a certain category of Azumaya algebras. For each (noncommutative) algebra R, we then consider the presheaf $\mathsf{Alg}(R, -)$, that sends each Azumaya algebra A to the set of algebra morphisms $R \to A$. We show that it is a sheaf for the so-called maximal flat topology, and consequently for all coarser Grothendieck topologies. The idea behind the sheaf $\mathsf{Alg}(R, -)$ is that it contains geometric information about the finite-dimensional representations of the algebra R. For example, if we take $A = \mathrm{M}_n(\mathbb{C})$, then $\mathsf{Alg}(R, \mathrm{M}_n(\mathbb{C})))$ is the set of n-dimensional representations of R, and the restriction morphism determined by a ring map $\mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_n(\mathbb{C})$ corresponds to base change. As approximations to the sheaf $\mathsf{Alg}(R, -)$ we define an Azumaya representation scheme $\mathrm{rep}_A(R)$ for each Azumaya algebra A. It is a scheme relative over $\mathrm{Spec}(C)$, with C the center of A, such that the sections $\mathrm{Spec}(C) \to \mathrm{rep}_A(R)$ correspond to the ring morphisms $R \to A$. Next, we will look at some properties of the toposes associated to the different Grothendieck topologies. For a certain family of Grothendieck topologies, we can compute the topos points in terms of UHF-algebras. Further, in some specific cases, we can find an equivalent description of the toposes by considering a category of sheaves on commutative rings, equipped with an action of some projective general linear group.

In Chapter 6, we will introduce a generalization of toposes, based on the noncommutative Heyting algebras from the work of Cvetko-Vah [CV19]. One of the differences with (commutative) Heyting algebras is that there is no single top element, but rather a family of top elements. We will replace the subobject classifier of the category of presheaves by some related noncommutative Heyting algebra $\mathbf{H}$, and then generalize the definition of Grothendieck topologies to this setting, as a kind of closure operator on $\mathbf{H}$. Afterwards, we give a definition of a sheaf in this case. The category of sheaves is then called a noncommutative topos. We finish with an example of a noncommutative topos that is not a topos in the usual sense: the category of complete directed graphs with a coloring of the edges (in 4 colors).

Bibliographical comments

This thesis is based on the following papers:

- Jens Hemelaer and Lieven Le Bruyn,
  Azumaya representation schemes,
  preprint, arXiv:1606.07885 (2016). [HLB16]
- Jens Hemelaer,
  Azumaya toposes,
  preprint, arXiv:1707.03814 (2017). [Hem17]
- Karin Cvetko-Vah, Jens Hemelaer and Lieven Le Bruyn,
  What is a noncommutative topos?,
  J. Algebra Appl. (2018), arXiv:1705.02831. [CVHLB18]
- Jens Hemelaer,
  An arithmetic topos for integer matrices,
  J. Number Theory (accepted 2019), arXiv:1806.01887. [Hem18a]
- Jens Hemelaer,
  Grothendieck topologies on posets,
  preprint, arXiv:1811.10039 (2018). [Hem18b]

Some parts of the thesis are copied verbatim from the papers, while other parts are presented in a new way. To be more precise:

- Chapter 1 is an introduction to topos theory that does not contain any original results.
- Chapter 2 is almost identical to [Hem18b].
- Chapter 3 is based on the description of Grothendieck topologies on the big cell in [Hem17], and contains some new ideas as well.
- Chapter 4 corresponds to the paper [Hem18a], some alternative proofs are given.
- Chapter 5 combines [HLB16] and [Hem17].
- Chapter 6 is almost identical to [CVHLB18].

Nederlandstalige samenvatting

In deze thesis bespreken we enkele toepassingen van topostheorie in niet-commutatieve meetkunde. Toposssen zijn een veralgemening van topologische ruimten, gedefinieerd als de categorie van schoven op een categorie, ten opzichte van een zogenaamde Grothendiecktopologie.

Eerst kijken we naar de categorie van verzamelingen met een actie van de natuurlijke getallen verschillend van nul (onder vermenigvuldiging). Dit is een topos die voorkomt in het werk van Connes en Consani in verband met hun Arithmetic Site. Een van de resultaten in dit deel is een classificatie van de subtoposssen met genoeg punten.

Vervolgens kijken we naar de de verzameling $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ van 2×2-matrices met gehele coëfficiënten, met determinant verschillend van nul. In plaats van de topos hierboven, bestuderen we nu de categorie van verzamelingen met een actie van $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ (onder vermenigvuldiging van matrices). We vinden enkele verbanden met getaltheorie. Zo is de verzameling van punten bijvoorbeeld een dubbel quotiënt dat ook verschijnt in het Langlandsprogramma en in de classificatie van torsievrije abelse groepen van rang 2. Verder is er een link met het vermoeden van Goormaghtigh.

We proberen steeds technieken te gebruiken die algemener toepasbaar zijn dan enkel in de twee gevallen hierboven. Om dit te bereiken geven we een classificatie van enkele families van Grothendiecktopologieën voor algemene partieel geordende verzamelingen.

Een andere toepassing van topostheorie die we behandelen, situeert zich in de studie van Azumaya-algebra's. We construeren families van Grothendiecktopologieën op de categorie duaal aan een zekere categorie van Azumaya-algebra's. Voor de corresponderende toposssen kunnen we in een aantal gevallen de punten bepalen in termen van UHF-algebras, of een alternatieve beschrijving geven van de topos via groepacties van een projectieve lineaire groep.

We sluiten de thesis af met een bespreking van een veralgemening van het concept topos, gebaseerd op niet-commutatieve Heytingalgebra's. De hoop is dat deze theorie meer inzicht zou kunnen geven in de niet-commutatieve aspecten van de bovengenoemde voorbeelden.

Chapter 1

Short introduction to topos theory

In this chapter, we give a quick introduction to topos theory, to settle the notation and to make the thesis more self-contained.

The definitions and results in this chapter can be found in standard works on topos theory, see for example Mac Lane–Moerdijk [MLM94], Johnstone [Joh02b] or the more recent book by Caramello [Car18]. We managed to keep this introduction short by only including the results that are directly relevant to the thesis. We assume that the reader is familiar with category theory, including for example universal properties, adjoint functors and 2-categories.

1.1 Basic notions

All categories $\mathcal{C}$ in the thesis will be *locally small*, i.e. for every two objects C, D the morphisms $C \to D$ form a set. If additionally the objects of $\mathcal{C}$ form a set, then $\mathcal{C}$ will be called a *small category*.

The set of morphisms from C to D will be denoted by $\mathcal{C}(C, D)$, unless stated otherwise. For $f \in \mathcal{C}(C, D)$, we say that C is the domain and D is the codomain.

Let $\mathcal{C}$ be a small category. For an object C in $\mathcal{C}$, a *sieve on* C is a family of morphisms

$$S = \{f : D \to C\} \tag{1.1}$$

with codomain C, such that $f \in S$ implies $f \circ g \in S$ (whenever composition is defined). A *Grothendieck topology* J is a certain kind of function assigning to each object C a collection of sieves on C, that we will call J*-covering sieves* (on C), or just *covering sieves* when J is clear from the context. In order for J to be a Grothendieck topology, it has to satisfy the following three axioms, see Mac Lane–Moerdijk [MLM94, III,3.2].

(GT1) The maximal sieve, consisting of all morphisms with codomain C, is a covering sieve.

(GT2) If $S \in J(C)$ and $f : D \to C$ is a morphism, then

$$f^{-1}(S) = \{g : E \to D \mid f \circ g \in S\} \tag{1.2}$$

is a covering sieve.

(GT3) If $S \in J(C)$ and R is a sieve on C such that $f^{-1}(R)$ is a covering sieve for any $f \in S$, then R is a covering sieve.

A *presheaf* $\mathcal{F}$ on $\mathcal{C}$ is a functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathsf{Sets}$ to the category of sets. So to each object C of $\mathcal{C}$ we associate a set $\mathcal{F}(C)$, and to each morphism $f : D \to C$ in $\mathcal{C}$ we associate a *restriction morphism* $f^* : \mathcal{F}(C) \to \mathcal{F}(D)$, such that $(f \circ g)^* = g^* \circ f^*$. A morphism of presheaves $\mathcal{F} \to \mathcal{G}$ is a natural transformation from $\mathcal{F}$ to $\mathcal{G}$ (as functors). For $\mathcal{F}$ and $\mathcal{G}$ presheaves, the set of morphisms $\mathcal{F} \to \mathcal{G}$ will always be denoted by $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$. The *category of presheaves* on $\mathcal{C}$ and presheaf morphisms between them is denoted by $\mathsf{PSh}(\mathcal{C})$.

For $\mathcal{F}$ a presheaf on $\mathcal{C}$ and C an object of $\mathcal{C}$, the elements $s \in \mathcal{F}(C)$ are called *sections*. Let S be a sieve on C, and take a family of sections $(s_f)_f$ indexed by the morphisms $f \in S$ with $s_f \in \mathcal{F}(D)$ for $f : D \to C$. Then the family $(s_f)_f$ is called a *matching family* if

$$g^*(s_f) = s_{(f \circ g)} \tag{1.3}$$

whenever the composition makes sense. To make this more concrete, note that for any family of morphisms

$$X = \{f_i : C_i \to C\} \tag{1.4}$$

we can consider the *sieve* S_X *generated by* X consisting of the morphisms $f_i \circ g$ for some $f_i \in X$ and g arbitrary. In this case, every matching family is uniquely given by choosing sections $s_i \in \mathcal{F}(C_i)$ such that $g^*(s_i) = h^*(s_j)$ for any commuting diagram

$$\begin{array}{ccc} D & \xrightarrow{\;g\;} & C_j \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle f_j} \\ C_i & \xrightarrow[\;f_i\;]{} & C \end{array} . \tag{1.5}$$

Every sieve is of course generated by the family of all morphisms it contains, but often there are more interesting choices of generators.

We say that $\mathcal{F}$ is a *J-sheaf*, or just *sheaf* if J is clear from the context, if a matching family $\{s_f\}_f$ for a covering sieve S on C can always be written as

$$s_f = f^*(s), \;\; f \in S \tag{1.6}$$

for a unique $s \in \mathcal{F}(C)$. We say that the sections s_f *glue* to a unique section s.

For each object C of $\mathcal{C}$, we can define the *representable presheaf*

$$\mathbf{y}C = \mathcal{C}(-, C). \tag{1.7}$$

Note that $\mathbf{y}$ defines a functor from $\mathcal{C}$ to the category $\mathsf{PSh}(\mathcal{C})$ of presheaves on $\mathcal{C}$, called the Yoneda embedding.

Theorem 1.1 (Yoneda Lemma, see Mac Lane [Mac71, p. 61])**.** *Let $\mathcal{C}$ be a locally small category, and let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. Then there is a natural isomorphism*

$$\mathrm{Hom}(\mathbf{y}C, \mathcal{F}) \simeq \mathcal{F}(C).$$

Proof. To a natural transformation $f : \mathbf{y}C \to \mathcal{F}$, we associate $f(\mathbf{1}_C) \in \mathcal{F}(C)$. The inverse is given by associating to $s \in \mathcal{F}(C)$ the natural transformation given by

$$(\mathbf{y}C)(D) \to \mathcal{F}(D), \qquad g \mapsto g^*(s) \tag{1.8}$$

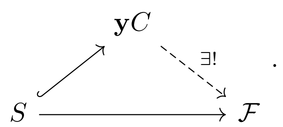
for each object D. Now check that the bijection is functorial in both C and $\mathcal{F}$. □

Corollary 1.2. *The Yoneda embedding is fully faithful (this explains the name).*

Proof. Take $\mathcal{F} = \mathbf{y}D$ in the Yoneda Lemma. $\square$

The Yoneda Lemma allows us to talk about *equivalence relations* in an arbitrary locally small category $\mathcal{C}$ with finite limits. We say that $f : R \to X \times X$ is an equivalence relation if $\mathbf{y}f : \mathbf{y}R \longrightarrow \mathbf{y}(X \times X) = \mathbf{y}X \times \mathbf{y}X$ is injective with as image an equivalence relation, whenever it is evaluated on an object C of $\mathcal{C}$. In particular f is a monomorphism.

We can use representable presheaves to reformulate the condition for $\mathcal{F}$ to be a J-sheaf. For C an object in $\mathcal{C}$, it is easy to see that sieves on C are the same as subpresheaves $S \subseteq \mathbf{y}C$, i.e. presheaves S with $S(D) \subseteq (\mathbf{y}C)(D)$ for each D. Moreover, matching families $(s_f)_{f \in S}$ are the same as presheaf morphisms $S \to \mathcal{F}$. So we can reformulate the condition for $\mathcal{F}$ to be a sheaf with the diagram



(1.9)

Here C is an object, $S \hookrightarrow \mathbf{y}C$ is a J-covering sieve on C, and $S \to \mathcal{F}$ is some presheaf morphism. We interpret a diagram as above in the following way: for each choice of "solid" morphisms, there exists a unique "dashed" morphism making the diagram commute.

As an immediate corollary of the reformulation, we get that $\lim_i \mathcal{F}_i$ is a J-sheaf, for any diagram of J-sheaves $(\mathcal{F}_i)_i$.

The full subcategory of $\mathsf{PSh}(C)$ consisting of the J-sheaves will be denoted by $\mathsf{Sh}(\mathcal{C}, J)$. We are now ready to state the most important theorem regarding Grothendieck topologies.

Theorem 1.3 (see Mac Lane–Moerdijk [MLM94, III.5])**.** *Let $\mathcal{C}$ be a small category and J a Grothendieck topology on $\mathcal{C}$. Then the inclusion functor i : $\mathsf{Sh}(\mathcal{C}, J) \to \mathsf{PSh}(\mathcal{C})$ has a left adjoint that preserves finite limits (the* sheafification functor*).*

Theorem 1.4 (Giraud's axioms, see Mac Lane–Moerdijk [MLM94, Appendix])**.** *Let $\mathcal{C}$ be a small category and let J be a Grothendieck topology on $\mathcal{C}$. Then the category $\mathsf{Sh}(\mathcal{C}, J)$ has the following properties:*

(a) it is locally small;

(b) it has a set of generators *T, i.e. a set of objects such that the morphism*

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \longrightarrow \prod_{\substack{f : \mathcal{A} \to \mathcal{F} \\ \mathcal{A} \in T}} \mathrm{Hom}(\mathcal{A}, \mathcal{G}) \tag{1.10}$$

(in each component given by precomposition) is injective;

(c) it has all finite limits (including a terminal object 1);

(d) it has all small coproducts (including an initial object 0);

(e) colimits are disjoint, i.e. the pullback of $\mathcal{F} \to \mathcal{F} \sqcup \mathcal{G} \leftarrow \mathcal{G}$ is 0;

(f) coproducts commute with pullbacks;

(g) the pushout diagram of an equivalence relation $R \to X \times X$

$$\begin{array}{ccc} R & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & Q \end{array} \tag{1.11}$$

is also a pullback diagram (we say that the quotient *Q is* effective*), and for any two morphisms $Y \to Z \leftarrow Q$ the pullback*

$$\begin{array}{ccc} R \times_Z Y & \longrightarrow & X \times_Z Y \\ \downarrow & & \downarrow \\ X \times_Z Y & \longrightarrow & Q \times_Z Y \end{array}$$

is a diagram of the same type ($R \times_Z Y$ is an equivalence relation on $X \times_Z Y$ with effective quotient $Q \times_Z Y$).

Conversely, any category with the above properties is equivalent to $\mathsf{Sh}(\mathcal{C}, J)$ for $\mathcal{C}$ some small category equipped with a Grothendieck topology J.

Definition 1.5. *A* Grothendieck topos *is a category equivalent to $\mathsf{Sh}(\mathcal{C}, J)$ for $\mathcal{C}$ some small category equipped with a Grothendieck topology J. Equivalently, a Grothendieck topos is a category satisfying the properties of Theorem 1.4.*

In this thesis, all toposes will be Grothendieck toposes, unless it is explicitly stated that we are talking about (the more general) elementary toposes.

Grothendieck toposes form a 2-category where

- the objects are Grothendieck toposes;
- the morphisms $f : \mathcal{T} \to \mathcal{T}'$ are adjunctions

$$\mathcal{T} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{T}' \tag{1.12}$$

  where the left adjoint f^* preserves finite limits (we will call these *geometric morphisms*);
- the 2-morphisms $f \Rightarrow g$ are natural transformations $f^* \Rightarrow g^*$ (we will call these *geometric transformations*).

There is a choice to be made for the direction of the morphisms and 2-morphisms. Our choice appears in Johnstone [Joh02a] and seems to be the most popular one. We denote by $\mathsf{Geom}(\mathcal{T}, \mathcal{T}')$ the *category of geometric morphisms* from $\mathcal{T}$ to $\mathcal{T}'$ and geometric transformations between them.

In this thesis, a *Grothendieck site* $(\mathcal{C}, J)$ is a small category $\mathcal{C}$ equipped with a Grothendieck topology J. In other texts, $\mathcal{C}$ might not necessarily be small in the definition of a Grothendieck site, or J might be a coverage or a (Grothendieck) pretopology.

The reason why we prefer Grothendieck topologies to the other approaches, is that the category of J-sheaves on $\mathcal{C}$ completely determine the Grothendieck topology J. We will make this more precise as follows. We say that a geometric morphism $f : \mathcal{T} \to \mathcal{T}'$ of toposes is a *geometric embedding* if the direct image part f_* is fully faithful. Equivalence classes of geometric embeddings are then called *subtoposes*. Now the following holds.

Proposition 1.6 (see Caramello [Car18, Theorem 1.3.35 and Theorem 1.3.36]). *Let $\mathcal{C}$ be a small category. Then there is a bijection between Grothendieck topologies on $\mathcal{C}$ and subtoposes of $\mathsf{PSh}(\mathcal{C})$, given by sending a Grothendieck topology J to the subtopos*

$$\mathsf{Sh}(\mathcal{C}, J) \longrightarrow \mathsf{PSh}(\mathcal{C})$$

defined by Theorem 1.3.

An important feature of Grothendieck toposes is that very different sites $(\mathcal{C}, J)$ and $(\mathcal{D}, J')$ can have equivalent toposes of sheaves. This is a point of view stressed in the work of Caramello, see for example [Car18]. In order to switch between sites, the following result is essential.

Theorem 1.7 (Comparison Lemma, see Mac Lane–Moerdijk[MLM94, p. 588]). *Let $\mathcal{C}$ be a small category with a Grothendieck topology J, and let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory. Now take J' to be the Grothendieck topology on $\mathcal{D}$ induced by J. Suppose that for each object C in $\mathcal{C}$ there is a J-covering sieve on $\mathcal{C}$ generated by objects of $\mathcal{D}$. Then there is an equivalence*

$$\mathsf{Sh}(\mathcal{C}, J) \simeq \mathsf{Sh}(\mathcal{D}, J'). \tag{1.13}$$

The Comparison Lemma can be used for switching to a larger site as well as switching to a smaller site.

1.2 Points of a topos and localic reflections

Definition 1.8. *Let $\mathcal{T}$ be a topos. Then a* (topos-theoretic) point of $\mathcal{T}$ *is a geometric morphism* $\mathsf{Sets} \to \mathcal{T}$*, and*

$$\mathsf{Pts}(\mathcal{T}) = \mathsf{Geom}(\mathsf{Sets}, \mathcal{T}) \tag{1.14}$$

is called the category of points of $\mathcal{T}$. If p is a point and $\mathcal{F}$ is an object of $\mathcal{T}$, then we say that $p^*\mathcal{F}$ is the stalk of $\mathcal{F}$ at p. *We say that $\mathcal{T}$* has enough points *if for any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ in $\mathcal{T}$, the property of being an isomorphism can be checked on stalks, i.e. the fact that*

$$p^*\varphi : p^*\mathcal{F} \to p^*\mathcal{G} \tag{1.15}$$

is an isomorphism for each point p, implies that φ is an isomorphism.

If $\mathcal{T} = \mathsf{Sh}(\mathcal{C}, J)$ then we say that J has enough points if and only if $\mathcal{T}$ has enough points.

Note that $\mathsf{Pts}(\mathcal{T})$ is in general not a small (or essentially small) category.[1] However, the following holds.

Proposition 1.9 (Johnstone [Joh02a, Corollary 2.2.12]). *If $\mathcal{T}$ has enough points, then there exists a* set *of points X, such that for any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ in $\mathcal{T}$,*

$$\varphi \text{ is an isomorphism} \Leftrightarrow p^*\varphi \text{ is an isomorphism, for all } p \in X. \tag{1.16}$$

[1] For example, let $\mathcal{C}$ be the opposite of the category of finite sets, and take $\mathcal{T} = \mathsf{PSh}(\mathcal{C})$. Then $\mathsf{Pts}(\mathcal{T})$ is the category of sets.

A set X as in the above proposition will be called a *separating set of points.*

Definition 1.10 (Space of points, based on Caramello [Car11, Definition 2.2])**.** *Let X be a set of points for a topos $\mathcal{T}$. For $p \in X$ and $U \hookrightarrow 1$ a subobject of the terminal object in $\mathcal{T}$, we say that U* contains *p if $p^*U = 1$. Then the sets*

$$\tilde{U} = \{p \in X \text{ such that } U \text{ contains } p\} \tag{1.17}$$

are the open sets of a topology on X, which is called the subterminal topology *(see [Car11, Theorem 2.3] for a proof).*

Note that, just like the open sets in a topological space, the subobjects $\mathsf{Sub}_{\mathcal{T}}(1)$ form a *frame*, i.e. a poset such that
(a) each family of elements $(x_i)_i$ has a supremum (or *join*) $\bigvee_i x_i$;
(b) each two elements y, z have an infimum (or *meet*) $y \wedge z$;
(c) these two operations are *distributive*: $(\bigvee_i x_i) \wedge y = \bigvee_i (x_i \wedge y)$.
A morphism of frames is a function preserving both the arbitrary supremums and finite infimums. The *category of frames* and frame morphisms will be denoted by Frm. The opposite category $\mathsf{Loc} = \mathsf{Frm}^{\mathrm{op}}$ is by definition the *category of locales.*

Each topological space can be seen as a locale through its frame of opens. So it makes sense to define a *point of a locale L* as a morphism of locales $\{*\} \to L$, where $\{*\}$ is the one point topological space. This is the same as a morphism of frames $\mathcal{O}(L) \to 2$, where $\mathcal{O}(L)$ is the frame associated to L, and 2 is the poset $\{0, 1\}$ with the obvious partial ordering. If $L = X$ is a topological space, then each $x \in X$ defines a point $x : \mathcal{O}(X) \to 2$ given by

$$x(U) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases} \tag{1.18}$$

In general, however, not all locale points of X are of this form.

The set of points of a locale L will be denoted by $\hat{L}$. The elements of $\mathcal{O}(L)$ are called the *opens* of L, and we say that $U \in \mathcal{O}(L)$ contains a point $p : \mathcal{O}(L) \to 2$ if $p(U) = 1$. In this way, the opens of L induce a topology on $\tilde{L}$.

Proposition 1.11 (see Mac Lane–Moerdijk [MLM94, Chapter IX])**.** *Let X be a topological space (seen as a locale). Then*
(a) the space of points $\hat{X}$ is sober*, i.e. every irreducible closed subset is the closure of a unique point;*
(b) the inclusion $X \subseteq \hat{X}$ induces an isomorphism of frames $\mathcal{O}(X) \simeq \mathcal{O}(\hat{X})$.
Moreover, if X is sober, then $\hat{X} = X$.

Because of this, $\hat{X}$ is often called the *sobrification of X.*

For $\mathcal{T}$ a topos, the *localic reflection $L\mathcal{T}$* is the locale dual to $\mathsf{Sub}_{\mathcal{T}}(1)$. Now we can state the most important result regarding the subterminal topology.

Theorem 1.12 (Based on Caramello [Car11, Theorem 2.3])**.** *If X is a separating set of points for $\mathcal{T}$, equipped with the subterminal topology, then its frame of opens is equivalent to $\mathsf{Sub}_{\mathcal{T}}(1)$.*

1.3 Examples

In the next chapter, we will study Grothendieck topologies on posets. A poset will always be interpreted as a category in the following way. The objects of the category are the elements of the poset, and there is a unique morphism $u \to v$ whenever $u \leq v$. With this interpretation, a category $\mathcal{C}$ is equivalent to a poset if and only if $\mathcal{C}(C, D)$ has at most one element for all objects C, D in $\mathcal{C}$.

1.3.1 Sheaves on a locale

For L a locale, we interpret the frame of opens $\mathcal{O}(L)$ in the above way, with a unique morphism $U \to V$ whenever $U \leq V$. We define a Grothendieck topology J_{loc} on $\mathcal{O}(L)$ by declaring a sieve $\{U_i \to U\}_i$ to be a covering sieve whenever $\bigvee_i U_i = U$. The topos

$$\mathsf{Sh}(L) = \mathsf{Sh}(\mathcal{O}(L), J_{\mathrm{loc}}) \tag{1.19}$$

is called the *topos of sheaves on* L. Clearly, the points of $\mathsf{Sh}(L)$ are in bijective correspondence to the locale points of L and the localic reflection of $\mathsf{Sh}(L)$ is L itself. More generally, there is a map

$$\mathsf{Loc}(L, L') \longrightarrow \mathsf{Geom}(\mathsf{Sh}(L), \mathsf{Sh}(L')) \tag{1.20}$$

and this map is an equivalence of categories, see Mac Lane–Moerdijk [MLM94, IX.5, Proposition 2]. Here $\mathsf{Loc}(L, L')$ is the category with as objects the morphisms $L \to L'$ (or equivalently, the frame morphisms $\mathcal{O}(L') \to \mathcal{O}(L)$), and as morphisms the natural transformations $f \Rightarrow g$, where the two frame morphisms $f, g : \mathcal{O}(L') \to \mathcal{O}(L)$ are interpreted as functors. We can reformulate this in the case $\mathsf{Loc}(1, L)$: there is a morphism (natural transformation) $p \to q$ if and only if every open set that contains p also contains q. This means $\mathsf{Loc}(1, L)$ is a poset, with partial ordering given by the specialization order on the space of points $\hat{L}$ of L.

Toposes that are equivalent to $\mathsf{Sh}(L)$ for some locale L, are called *localic toposes*. By Mac Lane–Moerdijk [MLM94, IX.5, Theorem 1], $\mathcal{T}$ is localic if and only if $\mathcal{T} \simeq \mathsf{Sh}(\mathcal{C}, J)$ with $\mathcal{C}$ a poset.

If $\mathcal{O}(L) = \mathcal{O}(X)$ for some topological space X, then we say that L is a *spatial locale*. In this case, $\mathsf{Sh}(L) = \mathsf{Sh}(X)$ has enough points, so by Theorem 1.12 we have $\mathcal{O}(L) = \mathcal{O}(\hat{L})$. Since L is spatial if and only if $\mathsf{Sh}(L)$ has enough points, we will sometimes confuse the terminology (saying that the locale has enough points, or that the topos is spatial).

1.3.2 Presheaf toposes

Let $\mathcal{T} \simeq \mathsf{PSh}(\mathcal{C})$ be a category of presheaves on a small category $\mathcal{C}$. Then we say that $\mathcal{T}$ is a *presheaf topos*. Presheaf toposes have enough points: for each object C of $\mathcal{C}$, the evaluation map $\mathcal{F} \mapsto \mathcal{F}(C)$ is the inverse image part of a geometric morphism $\mathsf{Sets} \to \mathsf{PSh}(\mathcal{C})$, i.e. it determines a point of $\mathsf{PSh}(\mathcal{C})$. In fact, the category of points for $\mathsf{PSh}(\mathcal{C})$ is opposite to the *category of pro-objects* $\mathcal{C}_{\mathrm{pro}}$ over $\mathcal{C}$, for which

- the objects are formal symbols $\varprojlim_i C_i$ for $(C_i)_i$ a cofiltered diagram; these formal symbols will be called *formal cofiltered limits*;
- the morphisms are given by

$$\mathcal{C}_{\mathrm{pro}}(\varprojlim_i C_i, \varprojlim_j C_j) = \varprojlim_j \varinjlim_i \mathcal{C}(C_i, C_j). \tag{1.21}$$

The point of $\mathsf{PSh}(\mathcal{C})$ associated to $\varprojlim_i C_i$ has inverse image part given by $\mathcal{F} \mapsto \varinjlim_i \mathcal{F}(C_i)$. The above was originally proved in Johnstone [Joh77, Proposition 7.13], under the assumption that $\mathcal{C}$ has fiber products. But as remarked by Gabber and Kelly in [GK15], this condition is unnecessary.

For $\mathcal{D}$ a small category, the *category of ind-objects* $\mathcal{D}_{\text{ind}}$ is the category for which

- the objects are formal symbols $\varinjlim_i D_i$ for $(D_i)_i$ a filtered diagram; these formal symbols will be called *formal filtered colimits*;
- the morphisms are given by

$$\mathcal{D}_{\text{ind}}(\varinjlim_i D_i, \varinjlim_j D_j) = \varprojlim_i \varinjlim_j \mathcal{D}(D_i, D_j). \tag{1.22}$$

Note that $(\mathcal{C}_{\text{pro}})^{\text{op}} \simeq (\mathcal{C}^{\text{op}})_{\text{ind}}$, and it will often be more convenient to compute $(\mathcal{C}^{\text{op}})_{\text{ind}}$ instead.

By Proposition 1.6, the Grothendieck topologies on $\mathcal{C}$ are in bijective correspondence with the subtoposes of $\mathsf{PSh}(\mathcal{C})$. Here the subtopos corresponding to a Grothendieck topology J is the topos of J-sheaves $\mathsf{Sh}(\mathcal{C}, J)$. The category of points for this subtopos $\mathsf{Sh}(\mathcal{C}, J)$ is the full subcategory of the category of points for $\mathsf{PSh}(\mathcal{C})$, consisting of the pro-objects $\varprojlim_i C_i$ such that for every covering sieve $\{D_j \to D\}_j$ and morphism $\varprojlim_i C_i \to D$, there is a j and a morphism $\varprojlim_i C_i \to D_j$ such that the diagram

$$\begin{array}{ccc} & D_j & \\ \nearrow & & \searrow \\ \varprojlim_i C_i & \longrightarrow & D \end{array} \tag{1.23}$$

commutes, see Gabber–Kelly [GK15, Proposition 1.4].

1.3.3 Sets with a monoid action

Let M be a monoid. Then we consider the category M-Sets of sets with a left M-action and equivariant maps between them. This will be called the *topos of M-sets*. Clearly, M-Sets is equivalent to $\mathsf{PSh}(M^{\text{op}})$, where M^{op} is the opposite monoid of M, interpreted as a category with one object, with morphisms given by the elements, and composition given by the multiplication in M^{op}.

Of course, the results from Example 1.3.2 can be applied to this special case, for example M-Sets always has enough points. And while the category of points for M-Sets can be very interesting, the localic reflection is not: the frame of subobjects of 1 is $\{0, 1\}$, corresponding to the trivial space with one point.

In Chapter 3 and 4 we will study the topos M-Sets for different monoids M.

1.3.4 Slice toposes

Let $\mathcal{C}$ be a category and take C in $\mathcal{C}$. Then the *slice category* $\mathcal{C}/C$ is defined as the category with

- as objects the morphisms $D \to C$ in $\mathcal{C}$; in other words, objects D of $\mathcal{C}$ equipped with a *structure morphism* to C;

- as morphisms the morphisms $D \to D'$ making the diagram

$$\begin{array}{ccc} D & \longrightarrow & D' \\ & \searrow \quad \swarrow & \\ & C & \end{array} \tag{1.24}$$

  commute.

For any topos $\mathcal{T}$ and object $\mathcal{F}$ of $\mathcal{T}$, the slice category $\mathcal{T}/\mathcal{F}$ is again a topos, see Mac Lane–Moerdijk [MLM94, IV.7]. This topos is called the *slice topos*. If $\mathcal{T} = \mathsf{Sh}(\mathcal{C}, J)$ and $\mathcal{F} = \mathbf{y}C$ for some C in $\mathcal{C}$, then we get $\mathcal{T}/\mathcal{F} = \mathsf{Sh}(\mathcal{C}/C, J)$, where a sieve $\{D_i \to D\}_i$ in $\mathcal{C}/C$ is a J-covering sieve if and only if the corresponding sieve in $\mathcal{C}$ is a J-covering sieve.

There is a geometric morphism $\pi : \mathcal{T}/\mathcal{F} \to \mathcal{T}$ where $\pi^*\mathcal{E}$ is the projection map $\mathcal{E} \times \mathcal{F} \to \mathcal{F}$. Moreover, π^* has a left adjoint (for this reason we call π an *essential geometric morphism*): it is given by the forgetful functor $\pi_! : \mathcal{T}/\mathcal{F} \to \mathcal{T}$.

For any point $p : \mathsf{Sets} \to \mathcal{T}$ and element $x \in p^*\mathcal{F}$, there is a point (p, x) of the slice topos $\mathcal{T}/\mathcal{F}$ such that $\pi(p, x) = p$. It is defined as follows. For every object $\mathcal{E}$ in $\mathcal{T}/\mathcal{F}$, there is a unique morphism $\mathcal{E} \to \mathcal{F}$ (where $\mathcal{F}$ has the identity as structure morphism). This induces a morphism $\xi_p : p^*\mathcal{E} \to p^*\mathcal{F}$, and now we set

$$(p, x)^*\mathcal{E} = \xi_p^{-1}(x). \tag{1.25}$$

In particular, if $p^*\mathcal{F} \neq \varnothing$ for all points p of $\mathcal{T}$, then the geometric morphism $\pi : \mathcal{T}/\mathcal{F} \to \mathcal{T}$ is surjective on points. In this case, $\mathcal{T}$ has enough points if and only if $\mathcal{T}/\mathcal{F}$ has enough points.

Analogously to the slice category $\mathcal{C}/C$, we can define the *coslice category* $C\backslash\mathcal{C}$ with

- as objects the morphisms $C \to D$ in $\mathcal{C}$; in other words, objects D of $\mathcal{C}$ equipped with a *structure morphism* from C;
- as morphisms the morphisms $D \to D'$ making the diagram

$$\begin{array}{ccc} & C & \\ & \swarrow \quad \searrow & \\ D & \longrightarrow & D' \end{array} \tag{1.26}$$

  commute.

In order to compute points of slice toposes, the following proposition will be useful.

Proposition 1.13. *Let $\mathcal{C}$ be a category and take C in $\mathcal{C}$. Then:*

$$(C\backslash\mathcal{C})_{\mathrm{ind}} \quad \simeq \quad C\backslash(\mathcal{C}_{\mathrm{ind}}). \tag{1.27}$$

Proof. A formal colimit of a filtered diagram $(C \to C_i)_{i \in I}$ on the left is sent to the induced morphism $C \to \varinjlim_{i \in I} C_i$ on the right. The morphisms agree, since

$$\begin{aligned} (C\backslash\mathcal{C})_{\mathrm{ind}}(\varinjlim_i C_i, \varinjlim_j C_j) &\simeq \varprojlim_i \varinjlim_j \, (C\backslash\mathcal{C})(C_i, C_j) \\ &\simeq (C\backslash(\mathcal{C}_{\mathrm{ind}}))(\varinjlim_i C_i, \varinjlim_j C_j). \end{aligned}$$

In the last natural isomorphism we use that $\lim_i C_i \to \lim_j C_j$ preserves the structure morphism from C if and only if in the corresponding family of maps $f_{ij} : C_i \to C_j$ each f_{ij} preserves the structure morphism from C. □

The category of points for $\mathsf{PSh}(\mathcal{C}/C)$ can now be computed as

$$\begin{aligned}((\mathcal{C}/C)_{\mathrm{pro}})^{\mathrm{op}} &\simeq ((\mathcal{C}/C)^{\mathrm{op}})_{\mathrm{ind}} \\ &\simeq (C\backslash\mathcal{C}^{\mathrm{op}})_{\mathrm{ind}} \\ &\simeq C\backslash(\mathcal{C}^{\mathrm{op}})_{\mathrm{ind}}.\end{aligned}$$

Note that $(\mathcal{C}^{\mathrm{op}})_{\mathrm{ind}}$ is the category of points for $\mathsf{PSh}(\mathcal{C})$. We finish the subsection with some examples.

Example 1.14.

- *Let $\mathcal{C}$ be the opposite of the category of finite sets and take C to be a singleton set. Then the category of points for $\mathsf{PSh}(\mathcal{C})$ is the category of sets. The category of points for $\mathsf{PSh}(\mathcal{C}/C)$ is the category of pointed sets.*
- *Let $\mathcal{C}$ be the opposite of the category of finitely presented commutative rings. Then the category of points for $\mathsf{PSh}(\mathcal{C})$ is the category of commutative rings. The category of points for $\mathsf{PSh}(\mathcal{C}/C)$ is the category of C-*algebras*: the objects are commutative rings R with a structure morphism $\varphi_R : C \to R$, the morphisms are ring morphisms $f : R \to S$ such that $f \circ \varphi_R = \varphi_S$.*

1.3.5 Equivariant sheaves

Let G be a discrete group acting continuously on a topological space X. Then the category $\mathsf{Sh}_G(X)$ of *G-equivariant sheaves* on X is a topos. Johnstone in [Joh02a, Example A.2.1.11(c)] gives a presentation $\mathsf{Sh}_G(X) \simeq \mathsf{Sh}(\mathcal{O}_G(X), T)$ defined as follows:

- $\mathcal{O}_G(X)$ is the category of open subsets of X, with morphisms $U \to V$ given by the elements $g \in G$ such that $g(U) \subseteq V$; composition is given by multiplication in G;
- T is the Grothendieck topology with as T-covering sieves the sieves
$$\{g_i : U_i \to U\}_i \tag{1.28}$$
such that $\bigcup_i g_i(U_i) = U$.

We will take this as a *definition* for the topos of G-equivariant sheaves on X.

Note that $\mathbf{y}X$ is a sheaf. It is easy to compute that the slice category $\mathcal{O}_G(X)/X$ is just $\mathcal{O}(X)$, and as a consequence $\mathsf{Sh}_G(X)/\mathbf{y}X \simeq \mathsf{Sh}(X)$. This defines a geometric morphism $\mathsf{Sh}(X) \to \mathsf{Sh}_G(X)$. The unique map $\mathbf{y}X \to 1$ is an epimorphism (even a surjection). This implies that $p^*(\mathbf{y}X) \neq \varnothing$ for every point $p : \mathsf{Sets} \to \mathsf{Sh}_G(X)$. So the geometric morphism $\mathsf{Sh}(X) \to \mathsf{Sh}_G(X)$ from the slice topos is surjective on points. For $x : \mathsf{Sets} \to \mathsf{Sh}(X)$ a point, the corresponding point $\pi(x) : \mathsf{Sets} \to \mathsf{Sh}_G(X)$ has inverse image part given by

$$\pi(x)^*\mathcal{F} = \varinjlim_{U \ni x} \mathcal{F}(U) \tag{1.29}$$

where the filtered colimit is taken over the diagram of open sets containing x, with the inclusion morphisms given by $1 \in G$. As a corollary, $\mathsf{Sh}_G(X)$ has enough points, with as separating set of points for example the elements of X.

From the above it is easy to reconstruct the category of points for $\mathsf{Sh}_G(X)$. Points are given by the elements of the sobrification $\hat{X}$. The morphisms $x \to y$ are given by the elements of

$$\begin{aligned}
&(\mathcal{O}_G(X)^{\mathrm{op}})_{\mathrm{ind}}(\varinjlim_{U\ni x} U, \varinjlim_{V\ni y} V) && (1.30)\\
&\simeq \varprojlim_{U\ni x} \varinjlim_{V\ni y} \mathcal{O}_G(X)^{\mathrm{op}}(U,V) && (1.31)\\
&\simeq \varprojlim_{U\ni x} \varinjlim_{V\ni y} \mathcal{O}_G(X)(V,U) && (1.32)\\
&\simeq \{\ g \in G\ :\ U \ni x \Rightarrow (\exists V \ni y)\ g(V) \subseteq U\ \} && (1.33)\\
&\simeq \{\ g \in G\ :\ U \ni x \Rightarrow U \ni g(y)\ \}. && (1.34)
\end{aligned}$$

With this description, it is easy to see that any morphism $g : x \to y$ in the category of points of $\mathsf{Sh}_G(X)$ can be factored as

$$x \xrightarrow{1} g(y) \xrightarrow{g} y. \qquad (1.35)$$

Here $g : g(y) \to y$ is an isomorphism with inverse $g^{-1} : y \to g(y)$.

This shows:

Proposition 1.15. *The points of* $\mathsf{Sh}_G(X)$ *are classified up to isomorphism by the* G*-orbits in the sobrification* $\hat{X}$*.*

We would also like to compute the localic reflection of $\mathsf{Sh}_G(X)$ here. Let $X/G \subseteq X$ be a set of representatives for the G-orbits in X. This is a separating set of points for $\mathsf{Sh}_G(X)$. By Theorem 1.12, it is enough to determine the subterminal topology on X/G. The subobjects of the terminal object $\mathcal{F} \hookrightarrow 1$ in $\mathsf{Sh}_G(X)$ are given by setting $\mathcal{F}(U)$ equal to either the empty set 0 or the one element set 1, according to the following axioms:

- $\mathcal{F}(U) = 1$ then $\mathcal{F}(V) = 1$ for all $V \subseteq U$;
- $\mathcal{F}(U) = 1$ then $\mathcal{F}(g(U)) = 1$ for all $g \in G$;
- $\mathcal{F}(U_i) = 1$ for all i then $\mathcal{F}(\bigcup_i U_i) = 1$.

There is a maximal set U such that $\mathcal{F}(U) = 1$ (take the union of all of them). Clearly, U is G-invariant, i.e. $g(U) = U$ for all $g \in G$. Conversely, for any G-invariant set U, we can set $\mathcal{F}(V) = 1$ if and only if $V \subseteq U$ and this satisfies the above axioms. So the subobjects $\mathcal{F} \hookrightarrow 1$ are in bijective correspondence to the G-invariant open sets, and $x^*\mathcal{F} = 1$ if and only if x is contained in the G-invariant open set associated to $\mathcal{F}$.

The induced topology on X/G is the quotient topology: a subset $Y \subseteq X/G$ is open if and only if $q^{-1}(Y)$ is open, where $q : X \to X/G$ is the quotient map.

Proposition 1.16. *The localic reflection of* $\mathsf{Sh}_G(X)$ *is the* quotient space X/G*, i.e.* X/G *with the quotient topology.*

Because taking the localic reflection of a topos is left adjoint to taking the category of sheaves on a locale, we get a geometric morphism $\mathsf{Sh}_G(X) \to \mathsf{Sh}(X/G)$. This might seem like an ideal quotient situation. In particular, one might hope to recover the isomorphism classes of points for $\mathsf{Sh}_G(X)$ by studying $\mathsf{Sh}(X/G)$ instead. However, it is often the case that two non-isomorphic points of $\mathsf{Sh}_G(X)$ are sent to the same point in $\mathsf{Sh}(G/X)$. The reason being that X/G is in general not T_0 (there are certain points that are not *topologically distinguishable* from each other, i.e. they are contained in exactly the same open sets).

Chapter 2

Grothendieck topologies on posets

When studying a category from a geometrical point of view, it is often desirable to have a concrete description of a certain Grothendieck topology on the category. This concrete description includes for example determining the points for such a Grothendieck topology, or answering whether or not the associated topos is coherent or subcanonical.

One example is the category $\mathcal{C} = \mathsf{Comm}_{\text{fp}}^{\text{op}}$, the opposite category of the category of finitely presented commutative rings. In Gabber–Kelly [GK15], very explicit descriptions are given for the points of various Grothendieck topologies.[1] As a demonstration of why this is useful, they point out an application to sheaf cohomology [GK15, Proposition 4.5]. For this category $\mathsf{Comm}_{\text{fp}}^{\text{op}}$, it is already an open problem to give a complete description for the points of the flat topology (some partial results are in [GK15, Lemma 3.3] and in Schröer [Sch17]).

However, if the category $\mathcal{C}$ is a poset, it is much easier to describe the Grothendieck topologies on it. The reason is that $\mathsf{PSh}(\mathcal{C})$ is equivalent to the category of sheaves on a topological space X, and Grothendieck topologies are in bijection to sublocales of $\mathcal{O}(X)$ (the locale of opens of X).

In this chapter, we exploit this fact to describe the Grothendieck topologies on posets. Of course, locales, posets and their interactions are a very well-studied topic, so all the necessary ingredients are already in the literature. Our contribution is translating these results to topos theory, and in this way extending the results of Lindenhovius [Lin14]. Later, we will discuss a classification of subtoposes of the Connes–Consani Arithmetic Site and some generalizations, where the strategy will be to reduce the problem to the localic case.

The underlying idea of this chapter is that a better understanding of Grothendieck topologies on posets can help us understanding Grothendieck topologies on more general categories. For example, in Le Bruyn [LB16, Proposition 2], it is shown that any Grothendieck topology on the monoid $\mathtt{C} = \mathbb{N}_+^{\times,\text{op}}$ (the underlying monoid for the Arithmetic Site of Connes and Consani), comes from

[1] Note that Gabber and Kelly work with sheaves on the category of separated schemes of finite type (and relative versions thereof). However, this is equivalent to sheaves on $\mathsf{Comm}_{\text{fp}}^{\text{op}}$ whenever the Grothendieck topology is finer than the Zariski topology, see Gabber–Kelly [GK15, Proof of Remark 2.4]. This follows from the Comparison Lemma.

a Grothendieck topology on a certain poset (the big cell). We will use this to study subtoposes of the Arithmetic Site in Section 3.4.

Similarly, in Chapter 5, Grothendieck topologies on the big cell are used to study Grothendieck topologies on the opposite of the category of Azumaya algebras and center-preserving algebra maps. Here an important part is determining which topologies on the big cell correspond to coherent subtoposes. We will show in Section 2.3 how to approach this problem for a very general class of posets.

2.1 Presheaves on a poset

We start by recalling some results mentioned in the previous chapter. First of all, we know that a category of sheaves on a poset is a localic topos. The category of *presheaves* on a poset in addition has enough points, so it is equivalent to $\mathsf{Sh}(X)$ for some topological space X. Moreover, we can assume that this space X is sober, and then it is uniquely determined. Indeed, the elements of X can be identified with the topos points of $\mathsf{PSh}(P)$ and the topology is the subterminal topology.

The category of points of $\mathsf{PSh}(P)$ is equivalent to $(P_{\text{pro}})^{\text{op}}$, so clearly it is itself equivalent to a poset. Up to equivalence, we can describe $(P_{\text{pro}})^{\text{op}}$ explicitly as the poset of *filters* on P under the inclusion relation. Here a filter is a nonempty, upwards closed, and downwards directed subset of P. We identify the elements of X with the filters on P, by sending a filter F to the formal cofiltered limit $\varprojlim_{p\in F} p$. Subterminal objects in $\mathsf{PSh}(P)$ are in bijection with the downwards closed subsets in P. For a downwards closed subset U of P, the corresponding open set for the subterminal topology is the set $\tilde{U}$ containing all filters intersecting $U \subseteq P$. In particular, if $\{a_i\}_{i\in I}$ is a set of generators of U, then the principal filters $p_i = \uparrow a_i$ are contained in U. So we can write the open sets as

$$(p_i)_{i\in I} = \{x \in X : \exists i \in I,\ p_i \leq x\} \subseteq X, \tag{2.1}$$

where $\leq$ denotes the partial order on the filters (i.e. inclusion of filters).[2]

Notice that filters on P are exactly the same thing as ideals on P^{op}, so this gives another point of view. In fact, this point of view is the most common in domain theory: for a poset P, the poset of ideals of P with the inclusion relation is called the *ideal completion*. So we can describe the category of points of $\mathsf{PSh}(P)$ as the ideal completion of P^{op}. Ideal completions are so-called *algebraic dcpo's*, with the principal ideals as *finite elements*, see Goubault-Larrecq [GL13, Proposition 5.1.46].

Definition 2.1 (Goubault-Larrecq [GL13, Section 5.1])**.** *Let X be a poset. Then for $a, b \in X$ we write*

$$a \ll b \tag{2.2}$$

if and only if $b \leq \sup(D)$ implies $\exists d \in D,\ a \leq d$, for any directed subset $D \subseteq X$ admitting a supremum.[3] *In this case, we say that a is* way below *b (this clearly implies $a \leq b$). We write*

$$\downarrow\!\!\!\!\downarrow x = \{y \in X : y \ll x\}. \tag{2.3}$$

[2]The notation $(p_i)_{i\in I}$ is inspired by the ideal notation for rings. Caveat: poset ideals are downwards closed, so $(p_i)_{i\in I}$ is almost never a poset ideal.

[3]A *directed subset* D is a subset for which any finite collection of elements in D has an upper bound in D. In particular, it is nonempty.

We say that $x \in X$ is finite *if $x \ll x$. We say that X is* algebraic *if for each $x \in X$ the set of finite elements smaller than x is directed with supremum x. We call a subset $B \subseteq X$ a* basis *if for each $x \in X$, the set $B \cap \downarrow x$ is directed with supremum x. We say that X is* continuous *if for each $x \in X$ the set $\downarrow x$ is directed with supremum x. We say that X is a* dcpo *if any nonempty directed subset of X admits a supremum.*

It is easy to show that a poset X is algebraic if and only if the set of finite elements is a basis, and it is continuous if there exists a basis. In particular, every algebraic poset is continuous, see Goubault-Larrecq [GL13, Lemma 5.1.23, Corollary 5.1.24].

Definition 2.2 (The Scott topology, see Goubault-Larrecq [GL13, Section 4.2])**.** *Let X be a poset. Then a subset $U \subseteq X$ is called Scott open if it is upwards closed and if for any directed subset $D \subseteq X$ with $\sup(D) \in U$ we have $U \cap D \neq \varnothing$.*

For algebraic posets, the Scott topology has as basis of open sets the sets

$$\uparrow x = \{y \in X : x \leq y\}$$

where x ranges over the finite elements of X [GL13, Theorem 5.1.27].

The specialization order *associated to the Scott topology is the original partial order on X.*[4]

Definition 2.3 (The dcpo of filters)**.** *Let P be a poset. Then the category of points for $\mathsf{PSh}(P)$ is equivalent to an algebraic dcpo X with as finite elements the principal filters on P.*

This dcpo will be called the dcpo of filters *on P. It comes equipped with the Scott topology as above. We will identify P^{op} with the poset of principal filters in X. Then the (Scott) open sets can be written as*

$$(a_i)_{i \in I} = \{x \in X : \exists i \in I,\ a_i \leq x\}$$

for some family $\{a_i\}_{i \in I}$ with each $a_i \in P^{\mathrm{op}}$.

We can now summarize the above results with the following corollary. The content of this corollary can be found e.g. in Amadio–Curien [AC98, Subsection 10.2], but no topos-theoretic language is used there. On the other hand, in Caramello [Car11, Subsection 4.2], the relation to topos theory is discussed and [Car11, Proposition 4.1] is very close to the corollary below.

Corollary 2.4 (See also Caramello [Car11, Proposition 4.1].)**.** *There is an equivalence of categories between:*

(a) the category of localic presheaf toposes and geometric morphisms between them (considered up to natural isomorphism);

(b) the category of algebraic dcpo's and Scott continuous maps between them.

Here Scott continuous *maps can be defined in two equivalent ways: either as monotonous maps preserving directed suprema, or literally as maps that are continuous for the Scott topology, see Vickers [Vic89, Theorem 7.3.1(iv)].*

[4] Here we define the specialization order on the points of a topological space as $x \leq y$ if and only if there is an inclusion of point closures $\mathrm{cl}(x) \subseteq \mathrm{cl}(y)$. The opposite convention is popular as well.

Proof. If a presheaf topos $\mathsf{PSh}(\mathcal{C})$ is localic, then $\mathcal{C}$ embeds (contravariantly) in the category of points of $\mathsf{PSh}(\mathcal{C})$, so $\mathcal{C} = P$ is a poset. Let X be the algebraic dcpo of filters on P. Conversely, any algebraic dcpo X is the category of points for $\mathsf{PSh}(F^{\mathrm{op}}) \simeq \mathsf{Sh}(X)$ where $F \subseteq X$ is the subset of finite elements.

The equivalence of categories now follows by the fact that the category of sober topological spaces and continuous maps is a full subcategory of the category of toposes and geometric morphisms (considered up to natural isomorphism), see Mac Lane–Moerdijk [MLM94, IX.3, Corollary 4] [MLM94, IX.5, Proposition 2]. □

2.2 Grothendieck topologies versus subsets

Let P be a poset, and let X be the dcpo of filters on P. Then by the results of the previous section $\mathsf{PSh}(P) \simeq \mathsf{Sh}(X)$, where X is equipped with the Scott topology. The Grothendieck topologies on P are in bijective correspondence with the subtoposes of $\mathsf{PSh}(P) \simeq \mathsf{Sh}(X)$, which in turn are in bijective correspondence with the sublocales of $\mathcal{O}(X)$, see Mac Lane–Moerdijk [MLM94, IX.5, Corollary 6]. In particular, the Grothendieck topologies with enough points correspond to the spatial sublocales, which are in bijective correspondence with the sober subspaces of X, see [MLM94, IX.3, Corollary 4].

If X is a sober space, then its sober subspaces are precisely the closed subspaces for a topology called the *strong topology*, see Keimel–Lawson [KL09, Corollary 3.5].

Definition 2.5 (Keimel–Lawson [KL09], Gierz et al. [GHK+03, Exercise V-5.31])**.** *Let X be a topological space. Then we say that $Y \subseteq X$ is* locally closed *if it is the intersection of an open set with a closed set. The locally closed sets clearly form a basis for a new topology, which we call the* strong topology *on X.*

If X with the original topology is T_0, then the strong topology is the one generated by the original topology and the downwards closed sets in the specialization order.

Corollary 2.6 (Corollary of Mac Lane–Moerdijk [MLM94, IX.3, Corollary 4])**.** *Let P be a poset and let X be its dcpo of filters (with the Scott topology). Then there is an adjunction*[5]

$$\mathcal{P}(X) \underset{S_{(-)}}{\overset{K_{(-)}}{\rightleftarrows}} \mathrm{GT}(P)^{\mathrm{op}} \tag{2.4}$$

between the poset $\mathcal{P}(X)$ of subsets of X (and inclusions), and the opposite of the poset $\mathrm{GT}(P)$ of Grothendieck topologies on P (and inclusions).

This adjunction restricts to an equivalence

$$\mathcal{V}(X) \;\simeq\; \mathrm{GT}_{\mathrm{ep}}(P)^{\mathrm{op}} \tag{2.5}$$

where $\mathcal{V}(X)$ denotes the full subcategory of subspaces that are closed for the strong topology, and $\mathrm{GT}_{\mathrm{ep}}(P)^{\mathrm{op}}$ denotes the full subcategory of Grothendieck topologies with enough points. Moreover, $\mathcal{V}(X)$ and $\mathrm{GT}_{\mathrm{ep}}(P)^{\mathrm{op}}$ are the images of $S_{(-)}$ and $K_{(-)}$, respectively.

[5] Adjunctions between categories that are posets are usually called Galois connections.

Proof. A similar adjunction is given in Mac Lane–Moerdijk [MLM94, IX.3, Corollary 4]. There it is proven that the two functors

$$\mathcal{O} \ : \ \mathsf{Top} \rightleftarrows \mathsf{Loc} \ : \ \mathsf{Pts} \tag{2.6}$$

are adjoint (with $\mathcal{O}$ the left adjoint). Here $\mathcal{O}(X)$ is the locale of opens of X, and $\mathsf{Pts}(L)$ is the space of points of L, with the subterminal topology. The adjoint functors are equivalences if we restrict to the category of locales with enough points on the left, and the category of sober spaces on the right (see loc. cit.).

In our situation, we can identify $\mathcal{P}(X)$ with the equivalence classes of embeddings $Y \hookrightarrow X$. Similarly, $\mathrm{GT}(P)^{\mathrm{op}}$ can be identified with the equivalence classes of locale embeddings $L \hookrightarrow \mathcal{O}(X)$. If $Y \hookrightarrow X$ is a topological embedding, one can check that $\mathcal{O}(Y) \hookrightarrow \mathcal{O}(X)$ is an embedding of locales. Conversely, if $L \hookrightarrow \mathcal{O}(X)$ is an embedding of locales, then $\mathsf{Pts}(L) \hookrightarrow X$ is a topological embedding of the spaces of points.

The adjunction between Top and Loc gives rise to a commuting diagram

$$\begin{array}{ccc} \mathsf{Loc}(\mathcal{O}(Y), L) & \xrightarrow{\sim} & \mathsf{Top}(Y, \mathsf{Pts}(L)) \\ \big\downarrow{\scriptstyle j\circ(-)} & & \big\downarrow{\scriptstyle j\circ(-)} \\ \mathsf{Loc}(\mathcal{O}(Y), \mathcal{O}(X)) & \xrightarrow{\sim} & \mathsf{Top}(Y, X) \end{array} , \tag{2.7}$$

for $i : Y \hookrightarrow X$ a topological embedding and $j : L \hookrightarrow \mathcal{O}(X)$ a locale embedding. This induces a natural bijection between the fibers of $j \in \mathsf{Loc}(\mathcal{O}(Y), \mathcal{O}(X)) \simeq \mathsf{Top}(Y, X)$ and this shows that $\mathcal{O}$ and Pts restrict to an adjunction between $\mathcal{P}(X)$ and $\mathrm{GT}(P)^{\mathrm{op}}$.

So we have proven the existence of the adjunction. The remaining statements follow directly from Mac Lane–Moerdijk [MLM94, IX.3], and the fact that adjunction is induced by the adjunction of [MLM94, IX.3, Corollary 4]. □

Take a sublocale $L \subseteq \mathcal{O}(X)$ corresponding to a Grothendieck topology J on P. Then from the above proof it follows that $S_J = \mathsf{Pts}(L) = \mathsf{Pts}(\mathsf{Sh}(P, J))$. Then using the criterion from Subsection 1.3.2 we can compute

$$S_J = \left\{ x \in X \ : \ p \le x \ \Rightarrow \ x \in \bigcap_{L \in \Omega_J(p)} \widetilde{L} \right\}. \tag{2.8}$$

Here $\Omega_J(p)$ denotes the set of J-covering sieves on p, and $\tilde{L}$ denotes the upwards closure of L in X (in other words, the Scott open set with as generators the generators of L).

Now take $Y \subseteq X$. The corresponding locale embedding is defined by the frame homomorphism $U \mapsto U \cap Y$. So the covering sieves L on p are the ones such that $\tilde{L} \cap (p)$ contains Y, where $\tilde{L}$ is the upwards closure of L in X (or in other words, the Scott open set corresponding to L). So we can write

$$K_Y(p) = \left\{ L \text{ sieve on } p \text{ such that } \tilde{L} \supseteq (p) \cap Y \right\}. \tag{2.9}$$

This gives a concrete description for the functors in Corollary 2.6.

2.3 Grothendieck topologies of finite type

A Grothendieck topology is called *of finite type* if every covering sieve contains a finitely generated covering sieve. In this section, we will relate Grothendieck topologies of finite type to coherent subtoposes and to patches (as introduced by Hochster in [Hoc69]).

We recall some definitions from Johnstone [Joh77, Subsection 7.3]. An object E in a topos is called *compact* if any epimorphic family with codomain E contains a finite epimorphic subfamily; the object E is called *coherent* if it is compact and any diagram $E' \to E \leftarrow E''$ of compact objects has a compact pullback.[6]

Definition 2.7 (See SGA 4 [sga72, Exposé vi, 2.4.5 and Définition 3.1])**.** *A* coherent topos *is a topos containing a full subcategory K of compact generators, that is moreover closed under finite limits. In this case, the objects of K are all coherent.*

A coherent geometric morphism *is a geometric morphism between coherent toposes such that the inverse image functor preserves coherent objects. We say that a subtopos of a coherent topos is a* coherent subtopos *if it is coherent and if the inclusion geometric morphism is coherent as well.*

As remarked by Johnstone in [Joh77, Remark 7.48], one family of coherent toposes is given by the categories of sheaves $\mathsf{Sh}(X)$ where X is a spectral space in the sense of Hochster [Hoc69].

Definition 2.8 (Hochster [Hoc69, Section 0 and 1])**.** *A topological space X is called a* spectral space *if*
(S1) it is sober;
(S2) it is compact;
(S3) its compact opens are closed under intersections and form a basis.
A continuous map $f : X \to Y$ between spectral spaces is called spectral *if the inverse image of a compact open is again compact open. A subspace $Y \subseteq X$ of a spectral space X is called a* spectral subobject *if it is a spectral space and the inclusion morphism is spectral.*

If X is a sober space, then $\mathsf{Sh}(X)$ is coherent if and only if X is spectral, see Johnstone [Joh77, Remark 7.48].[7]

If X is an algebraic dcpo with the Scott topology, then the open sets can all be written as $(p_i)_{i \in I}$ in the notation of (2.1); the elements p_i are called generators. Such an open set is clearly compact if and only if it can be generated by a finite set (in this case, we say that they are *finitely generated*). In particular, the compact opens form a basis. So X is a spectral space if and only if X itself is finitely generated and $(p) \cap (q)$ is finitely generated for any two opens (p) and (q) generated by a single element, see Priestley [Pri94, Theorem 2.4].

This is the case, for example, if the finite elements of X are a join-semilattice with a least element 0, because then $X = (0)$ and $(p) \cap (q) = (p \vee q)$. Another example is when X has only a finite number of finite elements. As a counter-example, take X the dcpo of filters on $(\mathbb{N}, \leq)$. Then X can be identified with

[6] In SGA 4 [sga72], "compact" is called "quasi-compact". In this thesis we will use the terminology "compact". Similarly, we use the word "compact" for (not necessarily Hausdorff) topological spaces such that every open cover admits a finite subcover.

[7] Note that Johnstone does not use the terminology "localic toposes" here, instead he calls them "toposes satisfying (SG)".

$\mathbb{N} \cup \{0\}$, with compact Scott open sets given by $\downarrow m = \{n \in \mathbb{N} : n \leq m\}$. So X is not compact.

In terms of the poset P this means:

Corollary 2.9 (See Johnstone [Joh77, Remark 7.48], Priestley [Pri94, Theorem 2.4])**.** *Let P be a poset and let X be its dcpo of filters. Then the following are equivalent:*

(a) $\mathsf{PSh}(P) \simeq \mathsf{Sh}(X)$ *is a coherent topos;*
(b) X *is a spectral space;*
(c) there is a finite list $t_1, \ldots, t_k \in P$ such that for all $p \in P$ there is a $t_i \geq p$, and the same holds for the subposets $\downarrow p \cap \downarrow q$ with $p, q \in P$.

Hochster's paper [Hoc69] is well-known for proving that every spectral space is homeomorphic to the spectrum of a commutative ring [Hoc69, Theorem 6]. This explains the terminology. In order to reach this result, Hochster introduced the so-called patch topology, which "classifies" spectral subobjects.

Definition 2.10 (Hochster [Hoc69])**.** *Let X be a spectral space. Then the* patch topology *is the new (finer) topology on X with as subbasis the compact opens of X and their complements. A closed set for the patch topology is called a* patch.

Proposition 2.11. *Let P be a poset and let X be its dcpo of filters. We assume that $\mathsf{PSh}(P) \simeq \mathsf{Sh}(X)$ is a coherent topos, or equivalently, that X is a spectral space. Let $S \subseteq X$ be a sober subspace. The following are equivalent:*

(a) $S \subseteq X$ *is a patch;*
(b) $S \subseteq X$ *is a spectral subobject;*
(c) K_S *is a Grothendieck topology of finite type;*
(d) $\mathsf{Sh}(S) \subseteq \mathsf{Sh}(X)$ *is a coherent subtopos.*

Proof. $(a) \Leftrightarrow (b)$. This is in Hochster [Hoc69, Section 2].

$(b) \Rightarrow (c)$. Let L be a K_S-covering sieve on $p \in P$. In other words,

$$\widetilde{L} \cap (p) \;\supseteq\; S \cap (p). \tag{2.10}$$

Now write $\widetilde{L} = (p_i)_{i \in I}$. Because $S \cap (p)$ is compact, we can find a finite subset $J \subseteq I$ such that the sieve corresponding to $(p_j)_{j \in J}$ is still a K_S-covering sieve.

$(c) \Rightarrow (d)$. Suppose that K_S be a Grothendieck topology on P of finite type. One can then show that $S \cap U$ is compact for every compact open $U \subseteq X$. The Yoneda embedding turns the compact opens $S \cap U \subseteq S$ into a generating set of compact objects, closed under pullbacks and containing the terminal object S (this shows it is closed under all finite limits). So $\mathsf{Sh}(S)$ is a coherent topos. We still have to show that the inclusion is coherent, i.e. the pullback functor $i^* : \mathsf{Sh}(X) \to \mathsf{Sh}(S)$ preserves coherent objects. It is enough to show that $i^*(\mathbf{y}U) = \mathbf{y}(U \cap S)$ is coherent for U a compact open of X. But $\mathbf{y}(U \cap S)$ is part of a generating family of compact objects, closed under finite limits. So it is coherent, see Definition 2.7.

$(d) \Rightarrow (b)$. Suppose that $\mathsf{Sh}(S) \subseteq \mathsf{Sh}(X)$ is a coherent subtopos. Then S is a spectral space by the remark above. We have to show that $U \cap S$ is compact open in S, for every compact open $U \subseteq X$. This follows by the assumption that the inclusion $\mathsf{Sh}(S) \subseteq \mathsf{Sh}(X)$ is coherent. □

We emphasize that the patch topology is finer than the (original) Scott topology, and coarser than the strong topology. In the situation of the above proposition: every closed set is a patch, and every patch is sober. Or: every closed subtopos is a coherent subtopos, and every coherent subtopos has enough points (special case of Deligne's completeness theorem).

Scott topology	Patch topology	Strong topology
"Closed" Grothendieck topologies	Grothendieck topologies of finite type	Grothendieck topologies with enough points
Closed subtoposes	Coherent subtoposes	Subtoposes with enough points

It is clear from the definition of spectral subobjects that all finite subsets $S \subseteq X$ are patches. More examples will be given in the following section.

We end with a note on terminology. The Scott topology is called the *localic topology* in the special case of the big cell in Le Bruyn [LB16] and Hemelaer [Hem17]. The strong topology on the big cell is called the *pcfb-topology* in [Hem17]. Moreover, the strong topology is sometimes called the *constructible topology* because it is the topology with the constructible sets as basis. Here a constructible set is defined as a finite union of locally closed sets. Sometimes in algebraic geometry, for example in Grothendieck [Gro66, Définition 9.1.2], a constructible set is defined differently, as a finite union of subsets of the form $U \cap V^c$, where U, V are so-called retrocompact opens. Then one could define the *constructible topology* on X (in the sense of Grothendieck) as the topology with the constructible sets (in the sense of Grothendieck) as basis, see for example [Sta19, Section 08YF]; in the case that X is spectral this is precisely the patch topology on X. If X is noetherian, then the strong topology and the patch topology agree, because in this case every open is compact. But in general the terminology "constructible topology" can be ambiguous, and it goes without saying that "strong topology" can mean different things as well (for example in Nerode [Ner59] it denotes what we call the patch topology[8]). Last but not least, in our situation the patch topology agrees with the so-called *Lawson topology* on (special kinds of) posets, see Gierz et al. [GHK+03] or Priestley [Pri94].

2.4 More explicit description

Let P be a poset and X its dcpo of filters, so $\mathsf{PSh}(P) \simeq \mathsf{Sh}(X)$. It is intuitively clear when a subset $S \subseteq X$ is closed for the Scott topology: this is if S is downwards closed and closed under directed suprema, see Gierz et al. [GHK+03, Remark II-1.4].

For the strong topology and the patch topology, it might be more difficult a priori to determine all closed sets.

Let $S \subseteq X$ be a subset closed for the strong topology. Take $s \notin S$. Then by definition we can find a (Scott) locally closed subset containing s, that does not intersect S. By shrinking the locally closed subset, we can assume it is of the form $(p) \cap \overline{s}$, where

$$\overline{s} = \{x \ : \ x \leq s\} \subseteq X \tag{2.11}$$

[8]This is mentioned by Priestley in [Pri94].

is the Scott closure of $\{s\}$. In other words, there exists a $p \in P$, $p \leq s$, such that S does not contain any x with $p \leq x \leq s$. This statement is logically equivalent to the following:

$$(\forall p \in P \text{ with } p \leq s,\ \exists x \in S \text{ with } p \leq x \leq s) \ \Rightarrow\ s \in S. \tag{2.12}$$

Conversely, it easy to see that any subset with the above property is closed for the strong topology. So the strong topology is in some sense a topology expressing "approximation from below".

Now we look at the patch topology. We apply an idea of Hochster [Hoc69] to our situation. Let W be the *Sierpinski space*[9]: it is the set $\{0,1\}$ with open sets $\varnothing, \{1\}, \{0,1\}$.[10] For X a topological space, the open subspaces of X are precisely the sets $f^{-1}(1)$ for some continuous map $f : X \to W$. Recall that X is spectral if and only if it is homeomorphic to a patch in $\prod_{i\in I} W$ for some index set I [Hoc69, Proposition 9].

Now we return to our case where P is a poset and X is its dcpo of filters *that we assume to be spectral* (for the Scott topology). Then we can say a little bit more. The finite elements of X are just P, but with the opposite ordering. So we can consider the continuous map

$$j\ :\ X \longrightarrow \prod_{p\in P} W \tag{2.13}$$

defined by

$$x \ \mapsto\ \begin{cases} 1 & \text{if } p \leq x, \\ 0 & \text{otherwise.} \end{cases} \tag{2.14}$$

in the component corresponding to $p \in P$. Here $\prod_{p\in P} W$ is itself the dcpo of filters on the poset P' of finite subsets of P, with the opposite of the inclusion relation. The product topology agrees with the Scott topology. In the next section we will have a look at $\mathsf{PSh}(P')$ in Example 2.22.

We saw before that, in our case, X is spectral if and only if it is compact and each $(p) \cap (q)$ is compact. The compact open sets in $\prod_{p\in P} W$ are the upwards closed sets generated by finitely many finite sets (we identify the elements of $\prod_{p\in P} W$ with subsets of P, the partial order is then inclusion of subsets). It is now straightforward to check that j is a spectral map turning X into a spectral subobject of $\prod_{p\in P} W$.

In the following we will always see X as a patch in $\prod_{p\in P} W$ in the way described above. In particular:

Lemma 2.12. *Let X be a spectral space and let $S \subseteq X$ be a subset. Then S is a patch in X if and only if it is a patch in $\prod_{p\in P} W$. Moreover, S is closed for the strong topology on X if it is closed for the strong topology on $\prod_{p\in P} W$.*

Proof. The last statement is clear.

We already mentioned that $j : X \longrightarrow \prod_{p\in P} W$ is spectral: if U is a compact open of $\prod_{p\in P} W$, then $j^{-1}(U)$ is again compact open. We want to prove that

[9] This is the notation of Hochster in [Hoc69]. Other symbols are S of $\mathbb{S}$, but both are already used in this thesis: S for a subspace of X or sometimes a sieve, $\mathbb{S}$ for the supernatural numbers.

[10] In comparison to the definition in [Hoc69], we swap 0 and 1. This is because Hochster uses the opposite definition of specialization order on a topology. We want that $0 \leq 1$ for the specialization order.

the patch topology on X is exactly the subspace topology with respect to the patch topology on $\prod_{p\in P} W$. It is enough to show that every compact open in X can be written as $j^{-1}(U)$ for some compact open U. Use that $j^{-1}((\{p\})) = (p)$ for each $p \in P$, and the fact that taking inverse images preserves unions and intersections. Here $(\{p\})$ denotes the compact open set in $\prod_{p\in P} W$ generated by the singleton set $\{p\}$. □

Theorem 2.13 (Priestley [Pri94, Theorem 2.1])**.** *For each index set I, the patch topology on $\prod_{i\in I} W$ agrees with the product topology, where W is equipped with the discrete topology.*

When identifying $\prod_{i\in I} W$ with the set of all set-theoretic functions $I \to \{0, 1\}$, the patch topology agrees with the topology of pointwise convergence.

Definition 2.14. *Let P be a poset and let X be its dcpo of filters. Let $(x_i)_i$ be a sequence of elements in X. Then we say that $(x_i)_i$* converges pointwise *to an element $x \in X$ if for all $p \in P$, there is a natural number N such that*

$$p \le x_i \;\Leftrightarrow\; p \le x \tag{2.15}$$

for all $i \ge N$.

Then by Lemma 2.12 and Theorem 2.13, a subset $S \subseteq X$ is a patch if and only if it is closed under pointwise convergence.

We say that $F \subseteq P$ is a *separating set (of finite elements)* if for all $p \in P$, we can write (p) as an intersection $(f_1) \cap \cdots \cap (f_k)$, with $f_1, \ldots, f_k \in F$. Then it is easy to see that the map $X \to \prod_{f\in F} W$ is injective, continuous and spectral. Moreover, Lemma 2.12 still holds if we replace P by F in the statement. To prove this we need that every compact open is the inverse image of a compact open in $\prod_{f\in F} W$, but this follows from F being a separating set.

We end with an application.

Proposition 2.15. *Let P be a countable poset and X its dcpo of filters. If X is spectral, then the patch topology on X is metrizable.*

Proof. We can embed X with the patch topology as a subspace of the space of set-theoretic functions $P \to \{0, 1\}$ with the pointwise convergence, which is a metric space if and only if P is countable.[11] □

Corollary 2.16. *Let P be a countable poset such that $\mathsf{PSh}(P)$ is a coherent topos. Then there is a metric space X with its closed subsets in natural bijection with the Grothendieck topologies of finite type on P.*

2.5 Cardinalities of sets of Grothendieck topologies

As an application of the explicit description from the previous section, we will compute the cardinalities of the sets of Grothendieck topologies on a poset

[11] The general result is that for a metric space Y, the space of continuous functions $X \to Y$ is metrizable if and only if X is hemicompact. Further, P with the discrete topology is hemicompact if and only if it is countable.

(resp. Grothendieck topologies with enough points, Grothendieck topologies of finite type, Grothendieck topologies giving rise to closed subtoposes).

Let P be a poset and let X be its dcpo of filters. We consider the topos $\mathsf{PSh}(P) \simeq \mathsf{Sh}(X)$. Note that in this case a *closed subtopos* is a subtopos $\mathsf{Sh}(Y) \subseteq \mathsf{Sh}(X)$ with $Y \subseteq X$ closed for the Scott topology.

We use the following notations:

cl	—	cardinality of set of closed subtoposes
coh	—	cardinality of set of coherent subtoposes
ep	—	cardinality of set of subtoposes with enough points
gt	—	cardinality of set of Grothendieck topologies
p	—	cardinality of P
x	—	cardinality of X

Proposition 2.17. *With the notations as above, suppose that X is an infinite spectral space. Then in the table*

			$2^{(2^{\mathbf{p}})}$
		$\mathbf{gt}$	$2^{\mathbf{cl}}$
	$2^{\mathbf{p}}$	$\mathbf{ep}$	$2^{\mathbf{x}}$
$\mathbf{x}$	$\mathbf{cl}$	$\mathbf{coh}$	$2^{\mathbf{p}}$

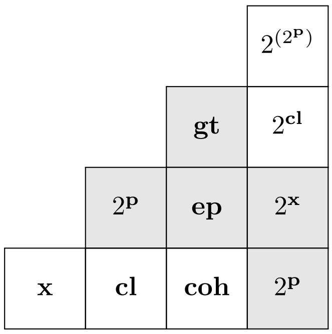
the cardinality in each box is less than or equal to the cardinalities in the boxes directly to the right of it and directly above it.

Proof. For the inequalities $\mathbf{x} \leq \mathbf{cl}$ use that all point closures are distinct (for the sober topology). The inequalities $\mathbf{cl} \leq \mathbf{coh} \leq \mathbf{ep} \leq 2^{\mathbf{x}}$ follow from the Scott topology being coarser than the patch topology, which is coarser than the strong topology, which is coarser than the discrete topology. The inequality $\mathbf{cl} \leq 2^{\mathbf{p}}$ follows from the fact that every Scott closed subset is determined by a set of elements in P. Further, each singleton $\{p\}$ with $p \in P$ is open for the strong topology, this shows $2^{p} \leq \mathbf{ep}$. For each $p \in P$, the set of finitely generated sieves on it has cardinality $\mathbf{p}$ (each sieve being uniquely determined by its finite set of generators). So $\mathbf{coh} \leq \mathbf{p}^{\mathbf{p}} = 2^{\mathbf{p}}$. An arbitrary Grothendieck topology is determined by a so-called nucleus, which is a function from the frame of opens to itself. So $\mathbf{gt} \leq \mathbf{cl}^{\mathbf{cl}} = 2^{\mathbf{cl}}$. Obviously, $\mathbf{ep} \leq \mathbf{gt}$. The inequalities in the last column follow directly from $\mathbf{p} \leq \mathbf{x}$ and the inequalities that we already proved. □

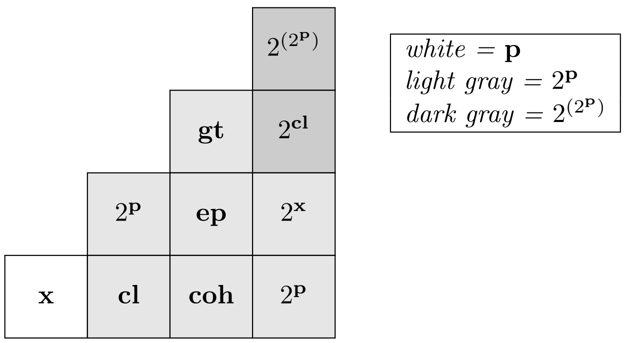
2.5.1 Artinian posets

First suppose that P is an *Artinian poset* (every subset has a minimal element).[12] This is the situation for which all Grothendieck topologies are explicitly described in Lindenhovius [Lin14]. In this case, all filters are principal, so $X = P^{\mathrm{op}}$. This also means that the open sets in X are just the upwards closed sets (or equivalently downwards closed sets in P). By [Lin16, Theorem 10.1.13] every Grothendieck topology is of the form K_S for some subset $S \subseteq P^{\mathrm{op}} = X$.[13]

Assume P infinite. Using a diagram like in Proposition 2.17: $\mathbf{p} = \mathbf{x}$ and in every gray box below, the cardinality is equal to $2^{\mathbf{p}}$.



Example 2.18 (Almost discrete posets)**.** *Let P be an infinite set containing a maximal element 1, with $p \le 1$ for each $p \in P$ as only relations. Then P is clearly Artinian and X is spectral. All subtoposes of $\mathsf{PSh}(P)$ have enough points. There are clearly $2^{\mathbf{p}}$ closed subspaces (equiv. closed subtoposes).*

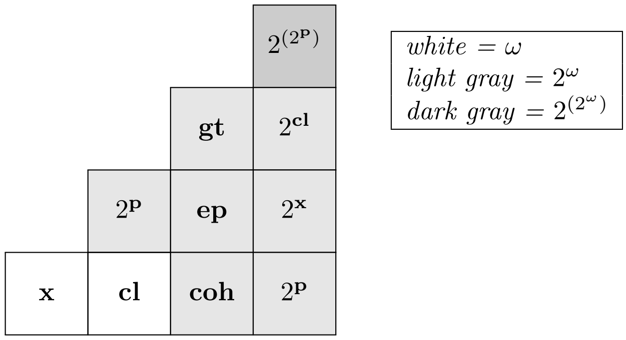


Example 2.19 (The natural numbers)**.** *Let P be the poset of natural numbers. In this case, there are countably many closed subtoposes (one for each natural number $n \in P$). So $\mathbf{p} = \mathbf{x} = \mathbf{cl}$. Further, like in the previous example, singletons are open in the patch topology. So the patch topology agrees with the discrete*

[12] A poset that is Artinian is sometimes said to be *well-founded*. The Artinian property is equivalent to the descending chain condition: every chain $p_1 \ge p_2 \ge p_3 \ge \dots$ eventually stabilizes.

[13] For subsets $S \subseteq P^{\mathrm{op}}$, the Grothendieck topology J_S from Lindenhovius [Lin14] and [Lin16] agrees with the Grothendieck topology K_S as in (2.9). The latter is inspired by the first one, with the only difference that it is defined for all subsets of X rather than subsets of $P^{\mathrm{op}} \subseteq X$.

topology, which shows $\mathbf{coh} = 2^{\mathbf{P}}$.

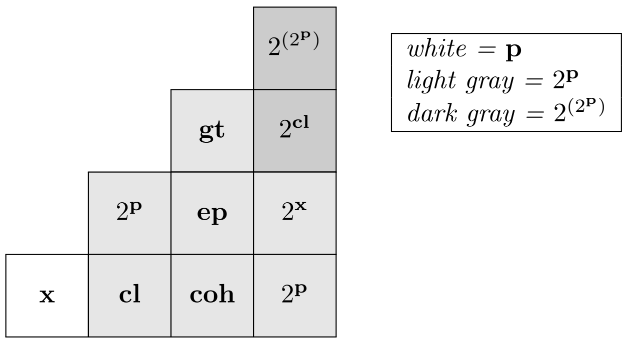


Here we use the notation ω *for the cardinality of the natural numbers. Similar computations can be done for* P *an arbitrary infinite ordinal.*

Example 2.20. *Let* P *be the poset of finite subsets of some infinite set* I*, with the inclusion relation. Then the cardinality of* P *is equal to the cardinality of* I*. For each subset* $I' \subseteq I$ *we can define the downwards closed set*

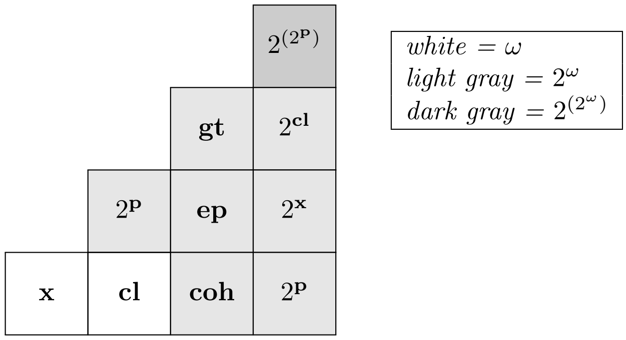
$$\Big\{\{i\} \;:\; i \in I'\Big\} \cup \{\varnothing\}. \tag{2.16}$$

This shows that there are $2^{\mathbf{P}}$ *open sets (so* $2^{\mathbf{P}}$ *closed sets as well).*



2.5.2 Other posets

Example 2.21. *Let* P *be the opposite of the poset of natural numbers. There is exactly one non-principal filter: the set* P *itself. So* $X = \{0, 1, 2, 3, \dots\} \cup \{\infty\}$*. Clearly* $\mathbf{p} = \mathbf{x} = \mathbf{cl}$*. Using Proposition 2.17 we then get* $\mathbf{gt} = \mathbf{ep} = 2^{\mathbf{P}}$*. All open sets in* X *are compact. So the patch topology agrees with the strong topology,* *and in particular* **coh** = **ep**.

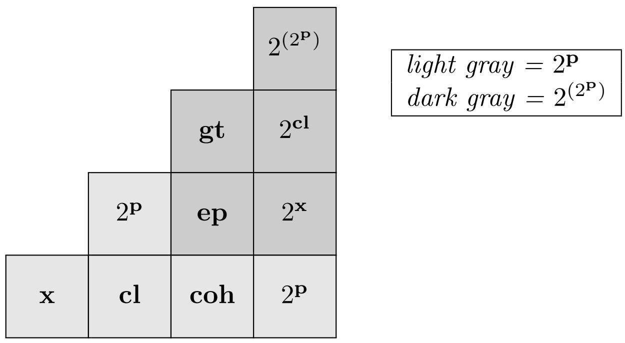


Example 2.22 (Power set)**.** *Let P be the poset of finite subsets of some infinite set I, with the opposite of the inclusion relation. Then $X = \mathcal{P}(I)$ with the inclusion relation. So* $\mathbf{x} = 2^{\mathbf{P}}$ *and by Proposition 2.17 this shows* $\mathbf{cl} = \mathbf{coh} = 2^{\mathbf{P}}$*. We still have to determine* **ep** *and* **gt**. *Note that every upwards closed set in X is closed for the strong topology (equiv. sober). Recall that an* antichain *in X is a subset such that all elements in it are pairwise incomparable. Sending an antichain to the upwards closed subset generated by it, is an injective operation, because the antichain can be recovered as the set of minimal elements.*[14]

In order to find how many antichains there are, we use a trick that seems to be well-known (at least in the case $I = \mathbb{N}$). Write $I = I_1 \sqcup I_2$ with $|I_1| = |I_2| = |I|$ and take a bijection $\psi : I_1 \to I_2$. Consider the set

$$A = \{x \in X \ : \ i \in x \Leftrightarrow \psi(i) \notin x\}. \tag{2.17}$$

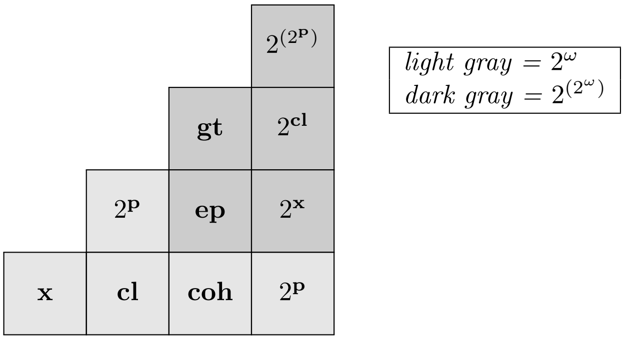
Note that A is an antichain: if $x \subseteq y$, $x, y \in A$, then $a \in x \Rightarrow a \in y$ but also $a \notin x \Rightarrow a \notin y$. So $x = y$. Clearly, $|A| = 2^{|I_1|} = 2^{\mathbf{P}}$, and each subset of A is again an antichain. This shows that there are at least $2^{(2^{\mathbf{P}})}$ antichains in X, and at least as many upwards closed sets. So we find $\mathbf{ep} = \mathbf{gt} = 2^{(2^{\mathbf{P}})}$.



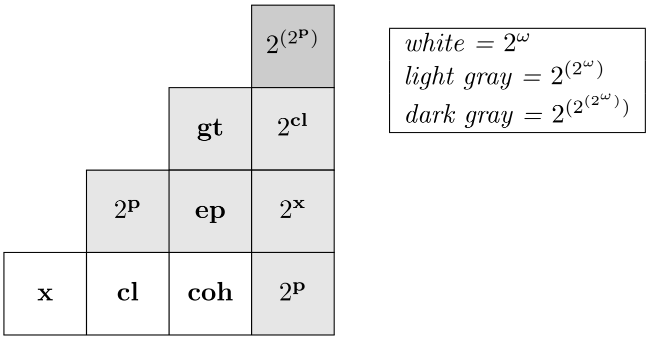
Example 2.23 (The big cell)**.** *Let P be the poset of positive natural numbers and the opposite of the division relation, so $m \leq n$ in P if and only if $n|m$. Then $X = \mathbb{S}$ is the poset of supernatural numbers under the division relation. The supernatural numbers will be introduced later. To determine* **gt**, **ep** *and* **coh**, *it is enough to know that there is a subposet $\mathcal{P}(I) \subset X$ where I is the set of prime*

[14] It is not surjective, consider for example the poset of real numbers and the upwards closed set $(0, +\infty)$.

numbers. So $\mathbf{x} = 2^{\mathbf{p}}$ *and this shows* $\mathbf{cl} = \mathbf{coh} = 2^{\mathbf{p}}$*. Each antichain in* $\mathcal{P}(I)$ *produces an antichain in* X*, so* $\mathbf{ep} = \mathbf{gt} = 2^{(2^\omega)}$.



Example 2.24. *Let* P *be the poset of nonnegative real numbers with the opposite partial order. Then the filters on* P *are* $[0,r]$ *or* $[0,r)$ *for some* $r \in \mathbb{R}$ *or* $[0,+\infty)$*. In* X *these filters will be denoted by* r *resp.* r_- *resp.* ∞*. Clearly* $\mathbf{x} = \mathbf{p} = 2^\omega$*. With a similar consideration, we see* $\mathbf{cl} = \mathbf{p}$*. A subbasis for the patch topology on* X *is given by the intervals* $[a,+\infty)$ *and their complements* $[0,a_-]$ *for* $a \geq 0$*. So a basis of the patch topology is given by the intervals* $[a,b_-]$ *for* $a < b$*,* $a,b \geq 0$*. Each open set* U *for the patch topology is then a disjoint union of open, closed and half-open intervals in* X*. Picking a rational number in each path-component shows that there are only countable many path-components in* U*. So to each open set we can associate a countable subset of the set of intervals in* X*. The set of intervals in* X *has cardinality* 2^ω*, so the set of countable subsets of it has cardinality* 2^ω *as well. We find that* $\mathbf{coh} = 2^\omega = \mathbf{x}$.



Remark 2.25. *For each of the inequalities resulting from Proposition 2.17, we have now given an example where the inequality is strict, with the exception of* $\mathbf{ep} \leq \mathbf{gt}$*. It is unknown to the author if it is possible to have* $\mathbf{ep} < \mathbf{gt}$*.*

Note that there are in each example at most three cardinalities (colors). This is no surprise: it is consistent with ZFC that there are at most three cardinalities $\mathbf{p} \leq \alpha \leq 2^{(2^{\mathbf{p}})}$ *(Generalized Continuum Hypothesis). So in ZFC it is not possible to construct an example with four colors or more.*

Chapter 3

The Connes–Consani Arithmetic Site

In [CC14], Connes and Consani introduced their *Arithmetic Site* as a framework for studying the Riemann Hypothesis. They start from two ingredients: the monoid $\mathbb{N}_+^\times$ of nonzero natural numbers under multiplication, and the *tropical semiring* $\bar{\mathbb{N}} = (\mathbb{N} \cup \infty, \inf, +)$. The semiring addition is defined by taking the minimum (infimum) of two numbers, whereas the semiring multiplication is given by adding numbers. There is an action of $\mathbb{N}_+^\times$ on $\bar{\mathbb{N}}$ by multiplication, and this action preserves both inf and $+$. In this way, $\bar{\mathbb{N}}$ becomes a semiring object in the topos $\mathbb{N}_+^\times$-Sets: the topos of sets with an action of $\mathbb{N}_+^\times$.[1]

By considering the tropical semiring as a structure sheaf, Connes and Consani brought together topos theory and tropical geometry. The interplay between the two is essential to their work. However, in this chapter we will devote all of our attention to the topos $\mathbb{N}_+^\times$-Sets *without* notion of structure sheaf.

We start the chapter by discussing the results from [CC14], [Con14], [CC16] by Connes and Consani and from [LB16] by Le Bruyn. In Section 3.2 we mention two alternative procedures to compute the category of points for the Arithmetic Site. In Section 3.3, we elaborate on the results of Hemelaer [Hem17, Section 2], in which different types of Grothendieck topologies on the big cell are classified. The results of this section are then used in Section 3.4 in order to describe subtoposes of the Arithmetic Site.

Some of the methods that we use here for the topos $\mathbb{N}_+^\times$-Sets, will later be generalized in Chapter 4, where we study the topos $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$. The Grothendieck topologies on the big cell will play a role in Chapter 5, because they are related to Grothendieck topologies on Azu (a suitable category of Azumaya algebras).

[1] In Connes–Consani [CC14], the monoid $\mathbb{N}_+^\times$ is written as $\mathbb{N}^\times$, and the topos $\mathbb{N}_+^\times$-Sets as $\widehat{\mathbb{N}^\times}$ (because it is the topos of presheaves on $\mathbb{N}^\times$ when interpreting the monoid as a one object category).

3.1 Preliminaries

We first discuss some previous results by Connes and Consani [CC14] [Con14] [CC16] and Le Bruyn [LB16].

Recall that Connes and Consani give a twofold description of the topos points for $\mathbb{N}_+^\times$-Sets:

Theorem 3.1 ([CC14], Theorem 2.2)**.**
(a) The category of points for $\mathbb{N}_+^\times$-Sets is equivalent to the category of nontrivial ordered subgroups of $\mathbb{Q}$, and arbitrary injective morphisms between them.
(b) There is a canonical bijection between the isomorphism classes of points for $\mathbb{N}_+^\times$-Sets and the double quotient $\mathbb{Q}_+^\times \backslash \mathbb{A}_f / \widehat{\mathbb{Z}}^\times$.

In the second part, $\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \prod_p \mathbb{Z}_p$ denotes the profinite integers, with $\widehat{\mathbb{Z}}^\times$ as subgroup of units. Further $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$ is the set of finite adeles, with

$$\mathbb{Q}_+^\times = \{q \in \mathbb{Q} : q > 0\} \tag{3.1}$$

acting by multiplication. For a proof of above theorem, see Connes [Con14] or Connes–Consani [CC16].

In this chapter, an important role is played by the Diaconescu covers from Le Bruyn [LB16]. Recall that a *Diaconescu cover* for a topos $\mathcal{T}$ is a geometric morphism

$$\pi : \mathcal{T}' \longrightarrow \mathcal{T} \tag{3.2}$$

that is open and surjective. *Open* means that the localic reflection

$$L\pi : L\mathcal{T}' \longrightarrow L\mathcal{T} \tag{3.3}$$

is an open map of locales. In particular, if $L\mathcal{T}$ and $L\mathcal{T}'$ are spatial, then it is enough to show that $(L\pi)(U)$ is open for every open $U \subseteq L\mathcal{T}'$. *Surjective* means that the inverse image part π^* is a faithful functor.

By Proposition 1.6, the subtoposes of $\mathbb{N}_+^\times$-Sets are in bijective correspondence to the Grothendieck topologies on $\mathtt{C} = \mathbb{N}_+^{\times,\mathrm{op}}$. Here a Grothendieck topology J corresponds to the subtopos $\mathsf{Sh}(\mathtt{C}, J) \subseteq \mathsf{PSh}(\mathtt{C}) \simeq \mathbb{N}_+^\times$-$\mathsf{Sets}$. Recall from Le Bruyn [LB16] that the *big cell*[2] $\mathtt{D}$ is the slice topos $\mathtt{C}/*$ over the unique object $*$ in $\mathtt{C} = \mathbb{N}_+^{\times,\mathrm{op}}$. As a poset, $\mathtt{D}$ is the set of positive natural numbers, with as partial ordering the opposite of the division relation (the direction of the morphisms is $m \to n$ for $m \leq n$ or equivalently $n \mid m$).

Proposition 3.2 (Le Bruyn [LB16, Proposition 2])**.** *Let J be a Grothendieck topology on $\mathtt{C}$. Let J_c be the Grothendieck topology on $\mathtt{D}$ such that $\{n_i \to n\}_i$ is a J_c-covering sieve if and only if $\{* \xrightarrow{n_i/n} *\}_i$ is a J-covering. Then the geometric morphism*

$$\mathsf{Sh}(\mathtt{D}, J_c) \longrightarrow \mathsf{Sh}(\mathtt{C}, J) \tag{3.4}$$

is a Diaconescu cover.

The *Steinitz numbers* of *supernatural numbers* are formal products

$$\prod_p p^{e_p} \tag{3.5}$$

[2] Terminology due to Conway [Con96].

over all primes p, with each $e_p \in \mathbb{N} \cup \{\infty\}$. Multiplication is defined in the obvious way. The positive natural numbers $\mathbb{N}_+$ are the supernatural numbers such that all exponents e_p are finite, and moreover $e_p = 0$ for almost all p. The supernatural numbers will be denoted by $\mathbb{S}$ (like in Le Bruyn [LB16]).

There is a multiplicative map $\widehat{\mathbb{Z}} \to \mathbb{S}$ sending a profinite number z to

$$\prod_p p^{e_p} \tag{3.6}$$

with p^{e_p} the maximal power of p dividing z (in particular $e_p = \infty$ if every power of p divides z). This map induces a bijection $\widehat{\mathbb{Z}}/\widehat{\mathbb{Z}}^\times \simeq \mathbb{S}$. Now we can rewrite the double qoutient as

$$\mathbb{Q}_+^\times \backslash \mathbb{A}_f / \widehat{\mathbb{Z}}^\times = \mathbb{N}_+^\times \backslash \mathbb{S} \tag{3.7}$$

(see Le Bruyn [LB16]). We say two supernatural numbers $s, s' \in \mathbb{S}$ are equivalent if they determine the same class in $\mathbb{N}_+^\times \backslash \mathbb{S}$, i.e. if there are natural numbers $n, m \in \mathbb{N}_+^\times$ such that $ns = ms'$. Recall also the suggestive notation from [LB16]

$$[\mathbb{S}] = \mathbb{N}_+^\times \backslash \mathbb{S} \tag{3.8}$$

for the equivalence classes of supernatural numbers, with $[s]$ the equivalence class corresponding to a supernatural number $s \in \mathbb{S}$.

Theorem 3.3 (Le Bruyn [LB16, Theorem 1])**.** *There is an equivalence of toposes*

$$\mathsf{PSh}(\mathtt{D}) \simeq \mathsf{Sh}(\mathbb{S}) \tag{3.9}$$

where $\mathbb{S}$ is seen as a topological space with open sets

$$(n_i)_{i \in I} = \{s \in \mathbb{S} \ : \ n_i \mid s \text{ for some } i \in I\} \tag{3.10}$$

for $\{n_i\}_{i \in I}$ a set of positive natural numbers indexed by I.

In other words, $\mathbb{S}$ can be identified with the dcpo of filters on the big cell (see the previous chapter).

Theorem 3.4 (Le Bruyn [LB16, Theorem 2])**.** *The Diaconescu cover*

$$\mathsf{PSh}(\mathtt{D}) \longrightarrow \mathsf{PSh}(\mathtt{C}) = \mathbb{N}_+^\times\text{-}\mathsf{Sets} \tag{3.11}$$

has as localic reflection the quotient map

$$\mathbb{S} \to [\mathbb{S}], \ s \mapsto [s]. \tag{3.12}$$

3.2 The category of points: alternative proofs

As discussed above, Connes and Consani proved that the category of points for $\mathbb{N}_+^\times$-Sets is equivalent to the category $\mathcal{L}_1$ with

- as objects the nontrivial ordered subgroups of $\mathbb{Q}$, where the ordering on $\mathbb{Q}$ is the standard one;
- as morphisms the injective morphisms of ordered groups.

We give two alternative proofs that $\mathcal{L}_1$ is the category of points for $\mathbb{N}_+^\times$-Sets. Neither is shorter than the proof given by Connes and Consani, and both make extensive use of topos theory. The advantage of the first proof is that it can be easily generalized to some other monoids. The second proof uses a description of $\mathbb{N}_+^\times$-Sets as a topos of equivariant sheaves, a result interesting on its own.

3.2.1 Using ind-categories

Let $\mathcal{L}_1^{\mathrm{fp}}$ be the full subcategory consisting of the objects of $\mathcal{L}_1$ that are isomorphic to $\mathbb{Z}$ (as ordered groups). Clearly, $\mathcal{L}_1^{\mathrm{fp}}$ is equivalent as a category to the monoid $\mathbb{N}_+^\times$. Further, any object $\mathcal{L}_1$ is a union (filtered colimit) of objects in $\mathcal{L}_1^{\mathrm{fp}}$, and

$$\mathcal{L}_1(\varinjlim_i \mathbb{Z}, \varinjlim_j \mathbb{Z}) \simeq \varprojlim_i \mathcal{L}_1(\mathbb{Z}, \varinjlim_j \mathbb{Z}) \quad (3.13)$$

$$\simeq \varprojlim_i \varinjlim_j \mathcal{L}_1(\mathbb{Z}, \mathbb{Z}). \quad (3.14)$$

This shows that $\mathcal{L}_1$ is the ind-category of $\mathcal{L}_1^{\mathrm{fp}} \simeq \mathbb{N}_+^\times$, and by the discussion in Subsection 1.3.2, this means $\mathcal{L}_1$ is the category of points for $\mathbb{N}_+^\times$-$\mathsf{Sets} \simeq \mathsf{PSh}(\mathbb{N}_+^{\times,\mathrm{op}})$.

Another result appearing in Connes–Consani [CC16] is the description of the points for $\mathbb{N}_0^\times$-Sets, with

$$\mathbb{N}_0^\times = \{n \in \mathbb{Z} : n \geq 0\} \quad (3.15)$$

(monoid law again given by multiplication). The category of points is now the category $\mathcal{L}_{\leq 1}$ with

- as objects the ordered subgroups of $\mathbb{Q}$;
- as morphisms the morphisms of ordered groups.

The alternative proof above also works in this case. We define $\mathcal{L}_{\leq 1}^{\mathrm{fp}}$ to be the full subcategory of $\mathcal{L}_{\leq 1}$ consisting of the objects of $\mathcal{L}_{\leq 1}$ isomorphic to $\mathbb{Z}$. Again, every object of $\mathcal{L}_{\leq 1}$ is a filtered colimit of objects in $\mathcal{L}_{\leq 1}^{\mathrm{fp}}$, and

$$\mathcal{L}_{\leq 1}(\varinjlim_i \mathbb{Z}, \varinjlim_j \mathbb{Z}) \simeq \varprojlim_i \varinjlim_j \mathcal{L}_{\leq 1}(\mathbb{Z}, \mathbb{Z}). \quad (3.16)$$

Further, there is an equivalence of categories $\mathcal{L}_{\leq 1}^{\mathrm{fp}} \simeq \mathbb{N}_0^\times$. So $\mathcal{L}_{\leq 1}$ is the category of points for $\mathbb{N}_0^\times$-Sets.

In Table 3.1 we give an overview of other toposes where this trick can be applied to determine the points. For the case of $\mathrm{M}_n(\mathbb{Z})$, it is slightly more difficult to prove that any subgroup $A \subseteq \mathbb{Q}^n$ can be written as a filtered colimit of groups isomorphic to $\mathbb{Z}^n$. It is easier in this case to first establish all groups $\mathbb{Z}^k$ for $k \in \{0, 1, 2, \dots, n\}$ as filtered colimits. This can be done by considering a chain $\cdots \to \mathbb{Z}^n \to \mathbb{Z}^n \to \dots$ with transition maps that are the identity on the first k components, and zero on the last $n - k$ components. Further, for any subgroup $A \subseteq \mathbb{Q}^n$, we can write A as the union of subgroups isomorphic to $\mathbb{Z}^k$ (with k the dimension of $A \otimes \mathbb{Q}$). Now we use the general result in category theory that

$$\varinjlim_i (\varinjlim_j A_{ij}) = \varinjlim_{(i,j)} A_{ij}. \quad (3.17)$$

In words: iterated filtered colimits are again filtered colimits, over a diagram with as objects the objects of the original diagrams.

The case $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$ will be studied in detail in a later chapter. For this topos there is already no known concrete description of the sets of points up to isomorphism, in the sense that for a family of points (given for example as subgroups $A_i \subseteq \mathbb{Q}^n$), it can be very difficult to determine whether or not there are two groups A_i, A_j within the family that are isomorphic to each other.

Monoid M	Category of points $\mathcal{L}$	Subcategory $\mathcal{L}^{\mathrm{fp}} \subseteq \mathcal{L}$
$\mathbb{N}_+^\times$	Nontrivial ordered subgroups of $\mathbb{Q}$, injective morphisms of ordered groups	Ordered subgroups isomorphic to $\mathbb{Z}$
$\mathbb{N}_0^\times$	Ordered subgroups of $\mathbb{Q}$, morphisms of ordered groups	Ordered subgroups isomorphic to $\mathbb{Z}$
$\mathbb{Z}_\pm = \{z \in \mathbb{Z} : z \neq 0\}$	Nontrivial subgroups of $\mathbb{Q}$, injective group morphisms	Subgroups isomorphic to $\mathbb{Z}$
$\mathbb{Z}$	Subgroups of $\mathbb{Q}$, group morphisms	Subgroups isomorphic to $\mathbb{Z}$
$\mathrm{M}_n^{\mathrm{ns}}(\mathbb{Z}) = \{a \in \mathrm{M}_n(\mathbb{Z}) : \det(a) \neq 0\}$	Subgroups $A \subseteq \mathbb{Q}^n$ such that $A \otimes \mathbb{Q} \cong \mathbb{Q}^n$, injective group morphisms	Subgroups isomorphic to $\mathbb{Z}^n$
$\mathrm{M}_n(\mathbb{Z})$	Subgroups $A \subseteq \mathbb{Q}^n$, group morphisms	Subgroups isomorphic to $\mathbb{Z}^n$

Table 3.1: Points for some toposes associated to monoids. The monoid law is in each case given by multiplication. The third column gives a subcategory $\mathcal{L}^{\mathrm{fp}} \subseteq \mathcal{L}$ with ind-category $\mathcal{L}$ and $\mathcal{L}^{\mathrm{fp}} \simeq M$.

3.2.2 Description as topos of equivariant sheaves

We already saw in the topos theory introduction that for *any* monoid M, the topos M-Sets has the one point topological space as localic reflection. So the localic reflection contains no information at all about the monoid M. We *can* find a separating set of points for M-Sets, because it is a presheaf topos, but the subterminal topology on this only has two open sets by Theorem 1.12 (originally Caramello [Car11, Theorem 2.3]).

We can however describe $\mathbb{N}_+^\times$-Sets as a topos of *equivariant* sheaves for a topological space with an action of a discrete group.

Proposition 3.5. *There is an equivalence of categories*

$$\mathbb{N}_+^\times\text{-}\mathsf{Sets} \;\simeq\; \mathsf{Sh}_{\mathbb{Q}_+^\times}(\mathbb{Q}_+^\times) \tag{3.18}$$

where $\mathbb{Q}_+^\times$ (as a discrete group) acts by multiplication on the topological space $\mathbb{Q}_+^\times$ with as open sets the sets U such that $n \cdot U \subseteq U$ for all $n \in \mathbb{N}_+^\times$.

Proof. Recall the site of definition for $\mathsf{Sh}_G(X)$ from Subsection 1.3.5. In our case, $\mathcal{O}_G(X)$ has

- as objects the sets $U \subseteq \mathbb{Q}_+^\times$ such that $n \cdot U \subseteq U$ for all $n \in \mathbb{N}_+^\times$;
- as morphisms $U \to V$ the elements $q \in \mathbb{Q}_+^\times$ such that $q \cdot U \subseteq V$.

The Grothendieck topology J has as J-covering sieves the sieves $\{q_i : U_i \to U\}_i$ such that $\bigcup_i q_i(U_i) = U$. Note that every open $U \subseteq \mathbb{Q}_+^\times$ is the union of open subsets of the form

$$\mathbb{N}_+^\times \cdot q = \{nq : n \in \mathbb{N}_+^\times\}, \tag{3.19}$$

for $q \in \mathbb{Q}_+^\times$. So by the Comparison Lemma (Theorem 1.7), we can replace the category $\mathcal{O}_G(X)$ above by the full subcategory $\mathcal{D}$ with

- as objects the elements $q \in \mathbb{Q}_+^\times$;
- as morphisms $q \to q'$ the elements $\alpha \in \mathbb{Q}_+^\times$ with $\alpha q \in \mathbb{N}_+^\times \cdot q'$.

The Grothendieck topology induced by J is the trivial topology (presheaf topology). Moreover, for any $q, q' \in \mathbb{Q}_+^\times$ there is an isomorphism $q \to q'$ given by q'/q, so it follows that $\mathcal{D}$ is equivalent to $\mathbb{N}_+^\times$. ☐

The above result is inspired by Le Bruyn [LB16], in which the space of supernatural numbers $\mathbb{S}$, with the topology as in [LB16, Theorem 1], is seen as a Diaconescu cover of $\mathbb{N}_+^\times$-Sets. Note that the topology on $\mathbb{Q}_+^\times$ introduced above is very similar.

In general, it is not true that a Diaconescu cover is surjective on points. Indeed: it is easy to see that the geometric morphism

$$p : \mathsf{Sets} \to M\text{-}\mathsf{Sets}, \quad p^*S = S \tag{3.20}$$

is a Diaconescu cover for *any* monoid M (it is open because the localic reflection is open, it is surjective because p^* is faithful).

The Diaconescu covers $\mathsf{Sh}(\mathtt{D}, J_c) \to \mathsf{Sh}(\mathtt{C}, J)$ satisfy the additional property that they are surjective on points. For $\mathsf{PSh}(\mathtt{D}) \to \mathsf{PSh}(\mathtt{C})$, this follows because $\mathtt{D} \simeq \mathtt{C}/*$, so $\mathsf{PSh}(\mathtt{D})$ is equivalent to the slice topos $\mathsf{PSh}(\mathtt{C})/\mathbf{y}(*)$ (and the geometric morphisms to $\mathsf{PSh}(\mathtt{C})$ are the same). Moreover, $p^*(\mathbf{y}(*)) \neq \varnothing$ for all points p, because the map $\mathbf{y}(*) \to 1$ is an epimorphism. So $\mathsf{PSh}(\mathtt{D}) \to \mathsf{PSh}(\mathtt{C})$ is surjective on points by the discussion in Subsection 1.3.4.

3.3 Grothendieck topologies on the big cell

In this section, we apply the results from the previous chapter to the big cell. First we classify the Grothendieck topologies on the big cell D that have enough points. Next we study the Grothendieck topologies that give rise to coherent subtoposes of $\mathsf{PSh}(\mathsf{D})$.

By Theorem 3.3, the dcpo of filters on the big cell is given by the supernatural numbers $\mathbb{S}$ with the division relation. The open sets for the Scott topology are

$$(n_i)_{i\in I} = \{s \in \mathbb{S} \ :\ \exists i,\ n_i \mid s\}. \tag{3.21}$$

We already computed the cardinalites **gt**, **ep**, **coh**, **cl** in Example 2.23 (amount of Grothendieck topologies resp. Grothendieck topologies with enough points resp. Grothendieck topologies of finite type resp. closed subspaces). We got the following diagram:

			$2^{(2^{\mathbf{P}})}$
		gt	$2^{\mathbf{cl}}$
	$2^{\mathbf{P}}$	**ep**	$2^{\mathbf{x}}$
x	**cl**	**coh**	$2^{\mathbf{P}}$

light gray = 2^{ω}
dark gray = $2^{(2^{\omega})}$

We will now explicitly describe different kinds of Grothendieck topologies on the big cell.

3.3.1 Grothendieck topologies for closed (open) subspaces

Determining Grothendieck topologies corresponding to closed (open) subspaces is easy and does not require the results of Chapter 2. So any techniques in this subsection are to be considered well-known, although we will use the language of Chapter 2.

For X a sober topological space, the closed and open subtoposes all have enough points, and taking points gives a bijection between the closed (open) subtoposes of $\mathsf{Sh}(X)$ on one hand and closed (open) subspaces of X on the other hand.

So the closed and open subtoposes of $\mathsf{PSh}(P)$ with P a poset, are in bijection with the closed resp. open subspaces of X, where X is the dcpo of filters on P. We can describe the corresponding Grothendieck topologies using Corollary 2.6.

The Grothendieck topology with enough points, corresponding to any subset $S \subseteq X$, is the topology K_S with:

$$L \text{ is a } K_S\text{-covering sieve on } p \ \Leftrightarrow\ \tilde{L} \supseteq S \cap (p), \tag{3.22}$$

for L a sieve on $p \in P$. Here $\tilde{L}$ is the upwards closure of L in X.

Back to the big cell. We first consider the closed subspaces $V \subseteq \mathbb{S}$. We can write

$$V = (n_i)^{\mathrm{c}}_{i\in I} = \{s \in \mathbb{S} \ :\ \nexists i,\ n_i \mid s\}. \tag{3.23}$$

For n in D, there are two possible situations:

- $\underline{n \in V.}$ Then $V \cap (n) \ni n$, so the only K_V-covering sieve on n is the maximal one.
- $\underline{n \notin V.}$ Then $V \cap (n) = \varnothing$, so any sieve on n is a K_V-covering sieve (including the empty sieve).

Note that the topos of sheaves $\mathsf{Sh}(\mathtt{D}, K_V) \simeq \mathsf{Sh}(V)$ is itself a presheaf topos: take the full subcategory $\mathtt{D}_V \subseteq \mathtt{D}$ with $\mathtt{D} \cap V$ as objects. Subsets of $\mathtt{D}_V$ are filters with respect to $\mathtt{D}_V$ if and only if they are filters with respect to $\mathtt{D}$. Now it is easy to see that the dcpo of filters on $\mathtt{D}_V$ can be identified with $V \subseteq \mathbb{S}$. Moreover, the Scott topology on V agrees with the subspace topology that we get from the Scott topology on $\mathbb{S}$. We conclude that $\mathsf{Sh}(\mathtt{D}, K_V) \simeq \mathsf{Sh}(V) \simeq \mathsf{PSh}(\mathtt{D}_V)$.

Now let us look at the Grothendieck topologies coming from (Scott) open subsets in $\mathbb{S}$. Take some open set $U = (n_i)_{i \in I}$. For each n in $\mathtt{D}$ there is a minimal sieve on n that is a K_U-covering sieve: the sieve $(n) \cap (n_i)_{i\in I} = (\mathrm{lcm}(n, n_i))_{i \in I}$. So the K_U-covering sieves are all sieves containing this minimal sieve.

Again, consider the full subcategory $\mathtt{D}_U$ with set of objects $\mathtt{D} \cap U$. Then the downwards closure of a filter in $\mathtt{D}_U$ is a filter in $\mathtt{D}$ (intersecting U). Conversely, a filter F on $\mathtt{D}$ intersecting U gives a filter $F \cap U$ on $\mathtt{D}_U$. This identifies the dcpo of filters on $\mathtt{D}_U$ with $U \subseteq \mathbb{S}$. Again the Scott topology on U agrees with the subspace topology coming from the Scott topology on $\mathbb{S}$. We conclude that $\mathsf{Sh}(\mathtt{D}, K_U) \simeq \mathsf{Sh}(U) \simeq \mathsf{PSh}(\mathtt{D}_U)$.

Example 3.6. *We give a few examples of closed or open subtoposes of* $\mathsf{PSh}(\mathtt{D})$.

- *For n in* $\mathtt{D}$*, (n) is homeomorphic to $\mathbb{S}$. This gives countably many ways of embedding $\mathbb{S}$ in itself as an open subspace. For Σ an infinite set of primes, take $s = \prod_{p \in \Sigma} p^\infty$. Then $\overline{s} = \{s' \ : \ s \mid s\}$ is also homeomorphic to $\mathbb{S}$. This gives 2^ω many ways of embedding $\mathbb{S}$ in itself as a closed subset.*
- *Let I be a countable set and let P be the poset of finite subsets of I, with the opposite of the inclusion relation. We already studied* $\mathsf{PSh}(P)$ *and its subtoposes in Example 2.22. Take $s = \prod_p p$ where the product is over all prime numbers (or infinitely many of them). Then for the point closure $\overline{s}$ we get $\mathtt{D}_{\overline{s}} = P$. So we can establish* $\mathsf{PSh}(P)$ *as a closed subtopos of* $\mathsf{PSh}(\mathtt{D})$*, in at least 2^ω many different ways.*
- *Take $s = p^\infty$ for some prime number p. Then $\mathtt{D}_{\overline{s}}$ is the set $\{p^k : k \in \mathbb{N}\}$, so it is isomorphic to the poset P of natural numbers with the opposite ordering, see Example 2.21. Now let V be the set of elements in $\mathbb{S}$ with at most one prime divisor (this is the complement of $(pq)_{p,q}$ where p, q run over pairs of distinct prime numbers). Then* $\mathsf{Sh}(V) \simeq \mathsf{PSh}(P')$*, where P' is a countable number of copies of P, all glued along the elements $0 \in P$.*

3.3.2 Grothendieck topologies of finite type

We saw in Proposition 2.11 that the Grothendieck topologies of finite type are in bijective correspondence with the patches in $\mathbb{S}$, i.e. the closed sets for the patch topology. Recall that we called $F \subseteq P$ a separating set if for all $p \in P$ we can write (p) as an intersection $(f_1) \cap \cdots \cap (f_k)$, with $f_1, \dots, f_k \in F$. In the case where $P = \mathtt{D}$ is the big cell, we can take F to be the subset of elements that are prime powers: for each n in $\mathtt{D}$, $(n) = (p_1^{e_1}) \cap \cdots \cap (p_k^{e_k})$, with $n = p_1^{e_1} \cdots p_k^{e_k}$ the prime factorization of n. Alternatively, we can take F to be the set of all positive natural numbers. These are the two examples to keep in mind below.

As discussed in Subsection 2.4, we then have a (spectral) embedding

$$\mathbb{S} \longrightarrow \prod_{f\in F} W \tag{3.24}$$

with W the Sierpinski space. We can identify $\prod_{f\in F} W$ with $\mathcal{P}(F)$, equipped with the Scott topology, as in Example 2.22. Because $\mathbb{S}$ is a spectral space, we saw that $\mathbb{S} \subseteq \mathcal{P}(F)$ is a patch. Then from Lemma 2.12 and Theorem 2.13 we get that

- $S \subseteq \mathbb{S}$ is a patch if and only if S is a patch for $\mathcal{P}(F)$;
- when identifying $\mathcal{P}(F)$ with the functions $F \to 2$, the patch topology is the topology of pointwise convergence;
- when identifying $\mathcal{P}(F)$ with $\prod_{f\in F}\{0,1\}$, the patch topology agrees with the product topology (w.r.t. discrete topology on $\{0,1\}$).

In particular $\mathcal{P}(F)$ with the patch topology is homeomorphic to the *Cantor set*. So it is metrizable, and $\mathbb{S}$ with the patch topology is metrizable as well, see Proposition 2.15.

Corollary 3.7. *Let $\{a_f\}_{f\in F}$ be a family of strictly positive real numbers indexed by a separating set F for* $\mathtt{D}$*, such that*

$$\sum_{f\in F} a_f < \infty. \tag{3.25}$$

For $s, s' \in \mathbb{S}$, define

$$F(s,s') = \{f \in F \ : \ f \textit{ divides one of } s, s' \textit{ but not the other}\}, \tag{3.26}$$

$$d(s,s') = \sum_{f\in F(s,s')} a_f. \tag{3.27}$$

Then d defines a metric *on $\mathbb{S}$ with the patch topology as induced topology. In particular, all metrics arising in this way are equivalent.*

Proof. Clearly, d is symmetric and $d(s,s') = 0 \Leftrightarrow s = s'$. The triangle inequality follows from $F(s,s'') \subseteq F(s,s') \cup F(s',s'')$.

Now interpret $\mathbb{S}$ as a subspace of the functions $F \to \{0,1\}$ with the topology of pointwise convergence. We have to show that the subspace topology agrees with the topology induced by d. Suppose we have a sequence $(s_n)_n$ of supernatural numbers converging pointwise to s. Then for each $\varepsilon > 0$, take a finite set $F_0 \subseteq F$ such that

$$\sum_{f\notin F_0} a_f < \varepsilon. \tag{3.28}$$

Now take an N such that, for $n > N$, s_n agree with s on all $f \in F_0$. Then for all $n > N$ we get

$$d(s_n, s) < \varepsilon. \tag{3.29}$$

Conversely, suppose that $d(s_n, s) \to 0$ for $n \to \infty$. Then clearly, for a certain $f \in F$, we can find N such that s_n and s agree on f for $n > N$. □

Proposition 3.8. *Take F, $\{a_f\}_{f\in F}$ and d as in Corollary 3.7. For $s, s' \in \mathbb{S}$, denote their greatest common divisor by $s \wedge s'$. Then:*

(a) $s \mid s' \mid s''$ implies $d(s,s'') = d(s,s') + d(s',s'')$;

(b) $d(s, s') = d(s, s \wedge s') + d(s \wedge s', s')$;
(c) $d(s, s') = d(1, s) + d(1, s') - 2d(1, s \wedge s')$.

Moreover, for $s \in \mathbb{S}$ and p a prime, let s_p be the biggest pth power dividing s. Suppose that F is the set of prime powers. Then:

$$d(1, s) = \sum_p d(1, s_p). \tag{3.30}$$

Suppose that F is the set of all positive natural numbers, and $n \mapsto a_n$ is multiplicative. Then:

$$1 + d(1, s) = \prod_p \Big(1 + d(1, s_p)\Big). \tag{3.31}$$

Proof. Straightforward to check. □

Example 3.9. *In the corollary above, take F to be the set of all natural numbers, and take $a_f = \frac{1}{f^2}$. In this case we can compute $d(1, \infty) = \zeta(2) - 1 = \frac{\pi^2 - 6}{6}$ and $d(1, p^\infty) = \frac{1}{p^2 - 1}$ (here ∞ denotes $\prod_p p^\infty$ where the product is over all primes). Further, $d(p, q) = \frac{p^2 + q^2}{p^2 q^2}$ for primes $p \neq q$.*

With the above criteria, it is in practice often easy to determine whether or not a subset $S \subseteq \mathbb{S}$ is a patch. We end the section with some examples of patches $S \subseteq \mathbb{S}$ and their associated Grothendieck topologies K_S of finite type. In each case, we will prove that S is a patch by showing that it is closed under pointwise convergence. An alternative approach would be to construct S explicitly from compact (Scott) open sets in $\mathbb{S}$ and their complements, using finite unions and arbitrary intersections. For the latter approach, see Hemelaer [Hem17, Example 2.7] (the examples we give here are the same as there, including the order in which they appear).

Example 3.10 (Examples of patches)**.**

(a) Finitely generated sieves. *Let $S = (m_1, \ldots, m_k)$ be a finitely generated open set in $\mathbb{S}$. Take a sequence $(s_n)_n$ with $s_n \in S$ for all n, and suppose that $s_n \to s$. We want to prove that $s \in S$. For each $i \in \{1, \ldots, k\}$, there is an N_i such that either $m_i \mid s_n$ for all $n > N_i$, or $m_i \nmid s_n$ for all $n > N_i$. Because $\{1, \ldots, k\}$ is finite, we are in the first case for at least one $i \in \{1, \ldots, k\}$. For this i we have $m_i \mid s$, so $s \in S$.*
For the description of K_S in this case, we refer to the previous subsection.

(b) Closed sets. *We can write each (Scott) closed set as*

$$S = (m_i)^{\mathrm{c}}_{i \in I} = \{s \in \mathbb{S} \ : \ \forall i \in I,\ m_i \nmid s\}. \tag{3.32}$$

If s_n is a convergent sequence in S, then $m_i \nmid s_n$ for all $i \in I$. But then if $s_n \to s$, then $m_i \nmid s$ for all $i \in I$, so $s \in S$.
For the description of K_S in this case, we refer to the previous subsection.

(c) Finite sets. *Let $S = \{x_1, \ldots, x_k\}$ be a finite set in $\mathbb{S}$, and let $(s_n)_n$ be a sequence of elements in S. Take $\varepsilon > 0$ strictly smaller than $d(x_i, x_j)$ for any $x_i \neq x_j$. If $(s_n)_n$ converges, then there is an N such that $d(s_n, s_m) < \varepsilon$ for all $n, m > N$. But then there is an i such that $s_n = x_i$ for all $n > N$. So $s = x_i \in S$.*
What are the K_S-covering sieves in this case? These are the sieves $\{m_j \to m\}_j$ such that for all $x_i \in S$ such that $m \mid x_i$, there is a j such that $m_j \mid x_i$.

In particular, for $S = \{\prod_p p^\infty\}$, we get the atomic topology*: the covering sieves are precisely the nonempty sieves.*

(d) Topologies from Hemelaer–Le Bruyn [HLB16]. *For Σ a set of primes, define $s_\Sigma = \prod_{p\in\Sigma} p^\infty$ and*

$$S_\Sigma = \{s \in \mathbb{S} \;:\; s_\Sigma \mid s\}. \tag{3.33}$$

Then S_Σ is a patch. Indeed, suppose $s_n \to s$ with $s_n \in S$ for all n. Then $p^k \mid s_n$ for all n and for all prime powers p^k with $p \in \Sigma$. So the same holds for s. This shows $s \in S_\Sigma$.
What are the K_S-covering sieves in this case? A sieve $\{m_i \to m\}_i$ is a covering sieve if and only if $\frac{m_i}{m}$ only has prime divisors in Σ, for at least one i. These topologies were introduced in Hemelaer–Le Bruyn [HLB16]. For $\Sigma = \varnothing$ this is called the minimal topology *(it agrees with the* chaotic topology / presheaf topology*). For Σ the set of all primes, this is called the* maximal topology *(it agrees with the atomic topology mentioned in the previous example).*

(e) Power set of the primes. *Take again $s_\Sigma = \prod_{p\in\Sigma} p^\infty$ for Σ a set of primes. Then*

$$2^{\mathcal{P}} = \{s_\Sigma \;:\; \Sigma \subseteq \mathcal{P}\} \tag{3.34}$$

is a patch, with $\mathcal{P}$ the set of all primes. Suppose $s_n \to s$ with $s_n \in 2^{\mathcal{P}}$. Suppose $p \mid s$. Then there is an N such that $p \mid s_n$ for all $n > N$. But then $p^k \mid s_n$ for all $n > N$, where k is arbitrary. In conclusion, $p \mid s \Rightarrow p^\infty \mid s$. This is the same as saying $s \in 2^{\mathcal{P}}$.
A sieve $\{m_i \to m\}_i$ is a K_S-covering sieve if the prime divisors of $\frac{m_i}{m}$ are also prime divisors of m, for at least one i.

(f) The ring spectrum Spec($\mathbb{Z}$). *For each prime number p, take*

$$s_p = \prod_{\substack{q\neq p\\ \text{prime}}} q^\infty. \tag{3.35}$$

Then $S = \{s_p : p \text{ prime}\} \cup \{\prod_p p^\infty\}$ is homeomorphic to Spec($\mathbb{Z}$)*, as observed by Le Bruyn [LB16]. We will show that it is a patch. Suppose that $s_n \to s$ for $s_n \in S$. As in Example (e), we can show that $p \mid s \Rightarrow p^\infty \mid s$. Now suppose that there are two primes $p \neq q$ such that $p \nmid s$ and $q \nmid s$. Then we can find a common N such that $p \nmid s_n$ and $q \nmid s_n$ for all $n > N$. This gives a contradiction. So there is at most one prime not dividing s. So $s \in S$.*
The K_S-covering sieves in this case are $\{m_i \to m\}_i$ with the property that whenever $p \nmid m$ with p prime, there is an m_i such that $p \nmid m_i$. Equivalently, $\{m_i \to m\}_i$ is a K_S-covering sieve if and only if $\gcd(m_i,\ i \in I)$ *divides m^k for some k. In ring theoretical terms, $\{m_i \to m\}_i$ is a covering sieve if and only if the radical of I is (m), where I is the ideal generated by the elements m_i.*

Lemma 3.11. *Let X be a finite T_0-space. Then X is homeomorphic to a patch in $\mathbb{S}$.*

Proof. It is enough to show that any finite poset can be embedded into $\mathbb{N}_+ \subseteq \mathbb{S}$, seen as a partial order by defining

$$n \leq_{\mathrm{d}} m \quad \Leftrightarrow \quad n \mid m. \tag{3.36}$$

We can do this by taking an arbitrary injection $f : X \hookrightarrow \mathcal{P}$ with $\mathcal{P}$ the set of primes. Now define

$$g : X \to \mathbb{N}_+, \quad g(x) = \prod_{y \leq x} f(y).$$

Then g is injective and preserves the partial order. □

Example 3.12 (Counterexample: the set of all primes)**.** *In $\mathbb{S}$, consider the subset S of all prime numbers. Then this is not a patch. Indeed, if $(s_n)_n$ is a sequence of primes, where each prime appears at most once, then $s_n \to 1$. The reason is that for each $m > 1$ there is an N such that $m \nmid s_n$ for all $n > N$. It is easy to show that the closure is $\{p \text{ prime}\} \cup \{1\}$.*

Another way to see that the set of primes is not a patch, is to calculate $d(1, p) = \frac{1}{p^2}$, where we take d as in Example 3.9. Then clearly $d(1, p) \to 0$ for $p \to \infty$.

Yet another way is to observe that the set of primes is not compact for the subspace topology (coming from the Scott topology on $\mathbb{S}$). Note that all patches are compact for the subspace topology by Definition 2.8 and Proposition 2.11 (originally Hochster [Hoc69, Section 2]).

3.3.3 Grothendieck topologies with enough points

As was shown in Corollary 2.6, the assignment $S \mapsto K_S$ gives a bijection between subspaces of $\mathbb{S}$ that are closed for the strong topology, and Grothendieck topologies with enough points. For a general subspace $S \subseteq \mathbb{S}$, the Grothendieck topology K_S has enough points as well and in fact $K_S = K_{\hat{S}}$, where $\hat{S}$ is the closure of S for the strong topology (or equivalently, the sobrification of S).

We will describe the subsets of $\mathbb{S}$ closed for the strong topology, using the criterion from Section 2.4. There we considered a general poset P and a subset $S \subseteq X$ with X the dcpo of filters for P. We then showed that S is closed for the strong topology if and only if the following holds:

$$(\forall p \in P \text{ with } p \leq s,\ \exists x \in S \text{ with } p \leq x \leq s) \ \Rightarrow\ s \in S. \tag{3.37}$$

If P is the big cell, then $X = \mathbb{S}$ and we can rewrite (3.37) as

$$(\forall n \in \mathbb{N}_+ \text{ with } n \mid s,\ \exists x \in S \text{ with } n \mid x \mid s) \ \Rightarrow\ s \in S. \tag{3.38}$$

Let $S \subseteq \mathbb{S}$ be a subset and let $\hat{S}$ be its closure for the strong topology. Denote the closure for the patch topology by $\mathrm{patch}(S)$ and write the upwards closure of S as $\uparrow S$. Then:

$$\hat{S} \subseteq \mathrm{patch}(S) \cap \uparrow S. \tag{3.39}$$

Example 3.13. *Consider the set $S = \{1\} \cup \uparrow\{2p : p \text{ prime}\}$. Then S is closed for the strong topology (it is the union of a patch and an upwards closed set). Further*

$$\uparrow S = \mathbb{S} \quad \textit{and} \quad \mathrm{patch}(S) = \{1\} \cup \uparrow\{2\}. \tag{3.40}$$

So $\hat{S} \subseteq \mathrm{patch}(S) \cap \uparrow S$ (but in this case $\hat{S} \neq \mathrm{patch}(S) \cap \uparrow S$).

3.4 Subtoposes of the Arithmetic Site

With the general approach of Chapter 2, we have described a variety of Grothendieck topologies on the big cell. But we were originally interested in subtoposes of the Arithmetic Site. Recall that the Arithmetic Site is the topos $\mathbb{N}_+^\times$-Sets of sets with an action of the monoid $\mathbb{N}_+^\times$ (we ignore the structure sheaf here). We are interested in its subtoposes. Note that there is an equivalence $\mathbb{N}_+^\times$-$\mathsf{Sets} \simeq \mathsf{PSh}(\mathbb{N}_+^{\times,\mathrm{op}})$.

We now apply the result of Le Bruyn [LB16, Proposition 2]: every Grothendieck topology $\mathcal{G}$ on $\mathtt{C} = \mathbb{N}_+^{\times,\mathrm{op}}$ can be lifted to a Grothendieck topology $\mathcal{G}_c$ on the big cell $\mathtt{D}$, in the following way. The covering sieves for $\mathcal{G}_c$ are the sieves $\{n_i \to n\}_{i\in I}$ such that

$$\{\tfrac{n_i}{n} : * \to *\}_{i\in I}. \tag{3.41}$$

is a $\mathcal{G}$-covering sieve. Clearly, $\mathcal{G} \neq \mathcal{G}'$ implies $\mathcal{G}_c \neq \mathcal{G}'_c$. Now let J be a Grothendieck topology on the big cell. Then $J = \mathcal{G}_c$ for some Grothendieck topology $\mathcal{G}$ on $\mathtt{C}$ if and only if

$$\{n_i \to n\}_{i\in I} \text{ is a covering sieve } \Leftrightarrow \{\tfrac{n_i}{n} \to 1\}_{i\in I} \text{ is a covering sieve.} \tag{3.42}$$

Note that for each Grothendieck topology $\mathcal{G}$ on $\mathtt{C}$, there is a natural geometric morphism $f : \mathsf{Sh}(\mathtt{D}, \mathcal{G}_c) \to \mathsf{Sh}(\mathtt{C}, \mathcal{G})$, see Le Bruyn [LB16, Proposition 2]. It is given by

$$(f^*\mathcal{F})(n) = \mathcal{F}(*) \tag{3.43}$$

for each $\mathcal{G}$-sheaf $\mathcal{F}$ on $\mathtt{C}$. From f^* we can compute f_*. The representable presheaf $\mathbf{y}(*)$ can be interpreted as the set of positive natural numbers, with left $\mathbb{N}_+^\times$-action given by multiplication. Let $\mathcal{F}$ be a $\mathcal{G}_c$-sheaf on $\mathtt{D}$. Then:

$$(f_*\mathcal{F})(*) \;\simeq\; \hat{\mathtt{C}}(\mathbf{y}(*), f_*\mathcal{F}) \;\simeq\; \hat{\mathtt{D}}(f^*\mathbf{y}(*), \mathcal{F}), \tag{3.44}$$

where we use the abbreviations $\hat{\mathtt{C}} = \mathsf{PSh}(\mathtt{C})$ and $\hat{\mathtt{D}} = \mathsf{PSh}(\mathtt{D})$. Further, write

$$(f_!\mathcal{F})(*) = \bigsqcup_{n\in\mathbb{N}_+^\times} \mathcal{F}(n) \tag{3.45}$$

where for $m \in \mathbb{N}_+^\times$ we define $m\cdot x$ to be the image of $x \in \mathcal{F}(n)$ along the restriction map $\mathcal{F}(n) \to \mathcal{F}(nm)$. It is easy to check that $f_!\mathcal{F}$ is a $\mathcal{G}$-sheaf whenever $\mathcal{F}$ is a $\mathcal{G}_c$-sheaf, and moreover that $f_!$ is left adjoint to f^*.

Corollary 3.14. *Let $\mathcal{G}$ be a Grothendieck topology on $\mathtt{C}$. Then there is an equivalence of toposes*

$$\mathsf{Sh}(\mathtt{D}, \mathcal{G}_c) \;\simeq\; \mathsf{Sh}(\mathtt{C}, \mathcal{G})/(\mathbf{y}(*)) \tag{3.46}$$

commuting with their natural geometric morphisms to $\mathsf{Sh}(\mathtt{C}, \mathcal{G})$. In particular,

$$\mathsf{Pts}(\mathsf{Sh}(\mathtt{D}, \mathcal{G}_c)) = \varphi^{-1}\Big(\mathsf{Pts}(\mathsf{Sh}(\mathtt{C}, \mathcal{G}))\Big) \tag{3.47}$$

$$\mathsf{Pts}(\mathsf{Sh}(\mathtt{C}, \mathcal{G})) = \varphi\Big(\mathsf{Pts}(\mathsf{Sh}(\mathtt{D}, \mathcal{G}))\Big). \tag{3.48}$$

Here we interpret $\mathsf{Pts}(\mathsf{Sh}(\mathtt{D}, \mathcal{G}_c))$ and $\mathsf{Pts}(\mathsf{Sh}(\mathtt{C}, \mathcal{G}))$ as subsets of $\mathbb{S}$ and $[\mathbb{S}]$ respectively, and $\varphi : \mathbb{S} \to [\mathbb{S}]$ is the projection map.

Corollary 3.15. *Let $\mathcal{G}$ be a Grothendieck topology on* $\mathtt{C}$*. Then:*

$$\mathcal{G} \text{ has enough points} \Leftrightarrow \mathcal{G}_c \text{ has enough points.}$$

Proof. This follows from Corollary 3.14 if we prove that

$$[s]^*\mathbf{y}(*) \neq \varnothing \tag{3.49}$$

for all points $[s] \in [\mathbb{S}]$ (see Subsection 1.3.4). Notice that the unique morphism $\mathbf{y}(*) \to 1$ is epi, and that $[s]^*$ preserves both epimorphisms and the terminal object. So $[s]^*\mathbf{y}(*) \to 1$ is surjective, from which the claim follows. □

Remark 3.16. *One can also use this description of* $\mathsf{Sh}(\mathtt{D}, \mathcal{G}_c)$ *as a slice topos in order to determine the functors* $f_!$*,* f^* *and* f_**.*

3.4.1 Subtoposes with enough points

We say that a subset $S \subseteq \mathbb{S}$ is *strongly invariant* under $\mathbb{N}_+^\times$ if both S and its complement S^{c} are $\mathbb{N}_+^\times$-invariant. A counter-example is $(n) \subseteq \mathbb{S}$ with $n \neq 1$. This subset is clearly $\mathbb{N}_+^\times$-invariant but the complement is not.

Lemma 3.17. *A subset $S \subseteq \mathbb{S}$ is strongly invariant under $\mathbb{N}_+^\times$ if and only if*

$$(n) \cap S = nS = \{ns : s \in S\} \tag{3.50}$$

for all $n \in \mathbb{N}_+^\times$.

Proof. The "only if" part is clear, so we only prove the "if" part. Suppose that $(n) \cap S = nS$ for all $n \in \mathbb{N}_+^\times$. Because $nS \subseteq S$ we see that S is invariant under $\mathbb{N}_+^\times$. Now take $s \notin S$ such that $ns \in S$. Clearly, $ns \in S \cap (n)$, but then it is also in nS, which gives a contradiction. So S^{c} is invariant under $\mathbb{N}_+^\times$ as well. □

To a Grothendieck topology $\mathcal{G}$ on $\mathtt{C}$, we can associate the subset $S \subseteq \mathbb{S}$ of points for the lifted Grothendieck topology $\mathcal{G}_c$. It turns out that this S is always strongly invariant under $\mathbb{N}_+^\times$.

Proposition 3.18. *The map $\mathcal{G} \mapsto \mathsf{Pts}(\mathtt{D}, \mathcal{G}_c)$ gives a bijective correspondence between the Grothendieck topologies on* $\mathtt{C}$ *with enough points, and the subsets of $\mathbb{S}$ that are upwards closed and strongly invariant under the action of $\mathbb{N}_+^\times$.*

Proof. We use Corollary 3.14. If $Y \subseteq [\mathbb{S}]$ is the set of points for $\mathcal{G}$, then

$$S = \{s \in \mathbb{S} \ : \ [s] \in Y\}$$

is the set of points for $\mathcal{G}_c$. From $[ns] = [s]$ we see that S is strongly $\mathbb{N}_+^\times$-invariant. Now assume $s \in S$ and $s \mid s'$. We have to prove that $s' \in S$. But for each $n \mid s'$, we know that $x_n = \mathrm{lcm}(n, s) \in S$. Now for each $n \mid s'$, we have $n \mid x_n \mid s'$ with $x_n \in S$. From (3.38) and the fact that S is closed for the strong topology, it follows that $s' \in S$.

Conversely, suppose that S is strongly $\mathbb{N}_+^\times$-invariant and upwards closed. Because it is upwards closed, it is in particular closed for the strong topology. Consider the Grothendieck topology K_S (this Grothendieck topology has enough points and the points are the elements of S). It remains to show that K_S comes from a Grothendieck topology on $\mathtt{C}$. We use the criterion (3.42).

$$
\begin{aligned}
\{n_i \to n\}_{i \in I} \text{ is a } K_S\text{-covering sieve} &\Leftrightarrow \forall s \in S \cap (n),\ \exists i \in I \text{ such that } n_i \mid s \\
&\Leftrightarrow \forall s \in nS,\ \exists i \in I \text{ such that } n_i \mid s \\
&\Leftrightarrow \forall s \in S,\ \exists i \in I \text{ such that } n_i \mid ns \\
&\Leftrightarrow \forall s \in S,\ \exists i \in I \text{ such that } \tfrac{n_i}{n} \mid s \\
&\Leftrightarrow \{\tfrac{n_i}{n} \to 1\}_{i \in I} \text{ is a } K_S\text{-covering sieve}
\end{aligned}
$$

□

3.4.2 Prime set topologies and duality

Definition 3.19. *Let $\mathcal{G}$ be a Grothendieck topology on $\mathtt{C}$. Suppose that $\mathcal{G}_c = K_S$ for $S \subseteq \mathbb{S}$ closed for the strong topology. Then we say that $\mathcal{G}$ is a* prime set topology *if S is generated by supernatural numbers of the form*

$$s_\Sigma = \prod_{p \in \Sigma} p^\infty \tag{3.51}$$

where Σ is a set of primes (a prime set*).*

Further, a prime set Σ is called significant *if $s_\Sigma \in S$, and* big *if the family $\{* \xrightarrow{p} *\}_{p \in \Sigma}$ generates a $\mathcal{G}$-covering sieve.*

Proposition 3.20. *Let $\mathcal{G}$ be a topology of finite type on $\mathtt{C}$. Then $\mathcal{G}$ is a prime set topology.*

Proof. Write S for the set of points of $\mathcal{G}_c$. Because $\mathcal{G}$ is of finite type, $\mathcal{G}_c$ is of finite type as well. So by Proposition 2.11, S is a patch (in addition to being upwards closed and strongly invariant, see Proposition 3.18). The claim is vacuous for S empty, so we can assume S to be nonempty. Take $s \in S$ and write s as a product

$$s \ = \ \left(\prod_{p \in \Sigma} p^\infty \right) \cdot s' \tag{3.52}$$

with Σ maximal. Because S is strongly invariant, s' has infinitely many prime divisors. And for every finite set of primes $\{q_1, \ldots, q_k\}$ not intersecting Σ, we can find s'' satisfying

$$\prod_{p \in \Sigma} p^\infty \ \mid \ s'' \ \mid \ s, \tag{3.53}$$

with $q_i \nmid s''$ for all $i = 1, \ldots, k$. Because S is a patch, this shows that $\prod_{p \in \Sigma} p^\infty$ is in S. It follows that $\mathcal{G}$ is a prime set topology. □

Remark 3.21. *The converse does not hold. Let S be the upwards closed set generated by the supernatural numbers of the form*

$$s_\Sigma = \prod_{p \in \Sigma} p^\infty \tag{3.54}$$

with Σ infinite. Then clearly S is strongly invariant under $\mathbb{N}_+^\times$, so we can find a Grothendieck topology $\mathcal{G}$ on $\mathtt{C}$ with $\mathcal{G}_c = K_S$. But S is not a patch, so $\mathcal{G}$ is not of finite type.

Proposition 3.22. *Let $\mathcal{G}$ be a prime set topology on $\mathtt{C}$ and let S be the set of points of $\mathcal{G}_c$.*

(a) If Σ is big and $\Sigma \subseteq \Sigma'$ then Σ' is big as well.

(b) If Σ is significant and $\Sigma \subseteq \Sigma'$ then Σ' is significant as well.

(c) Σ is significant if and only if the complement Σ^c is not big.

(d) Σ is big if and only if the complement Σ^c is not significant

(e) Let $L = \{* \xrightarrow{n_i} *\}_{i\in I}$ be a sieve on $*$ in $\mathtt{C}$. Then L is a covering sieve if and only if

$$L \subseteq \{* \xrightarrow{p} *\}_{p\in\Sigma} \quad \Rightarrow \quad \Sigma \text{ is big.} \tag{3.55}$$

Proof. (a) and (b) are straightforward. (c) and (d) are logically equivalent, so we only prove (c). Suppose that Σ is significant, so $s_\Sigma \in S$. We want to show that Σ^c is not big. If it were, then $L = \{p \to 1\}_{p\notin\Sigma}$ would be a K_S-covering sieve. But L does not contain s_Σ, a contradiction. Conversely, assume that Σ^c is not big; we want to show that Σ is significant. It is enough to show that any K_S-covering sieve $L = \{n_i \to 1\}_{i\in I}$ contains s_Σ. If L does not contain s_Σ, then for each n_i we can find a $p \mid n_i$ with $p \nmid s_\Sigma$ (in other words, with $p \notin \Sigma$). But this shows that $L \subseteq \{p \to 1\}_{p\notin\Sigma}$, so the latter is a K_S-covering sieve as well. It follows that Σ^c is big, a contradiction.

(e) The "only if" direction is clear, so we only prove the "if" direction. Assume that $L = \{* \xrightarrow{n_i} *\}$ satisfies property (3.55); we need to show that L is a covering sieve. It is enough to show that the lifted sieve $\{n_i \to 1\}_{i\in I}$ on $\mathtt{D}$ contains all s_Σ for Σ significant. If it does not, then as in the proof for (c), then we can find a significant Σ such that $L \subseteq \{p \to 1\}_{p\notin\Sigma}$ (same technique as in the proof of (c)). We know from (c) that Σ^c is not big, but this contradicts (3.55). □

Proposition 3.23. *Let Φ be a property on prime sets, such that Σ satisfying Φ and $\Sigma \subseteq \Sigma'$ implies that Σ' satisfies Φ.*

(a) There is a prime set topology $\mathcal{G}$ on $\mathtt{C}$ such that Σ satisfies Φ if and only if it is significant (with respect to $\mathcal{G}$).

(b) There is a prime set topology $\mathcal{G}$ on $\mathtt{C}$ such that Σ satisfies Φ if and only if it is big (with respect to $\mathcal{G}$).

Proof. (a) Consider the upwards closed set $S \subseteq \mathbb{S}$ generated by the supernatural numbers s_Σ with Σ satisfying Φ. Then S is strongly invariant under $\mathbb{N}_+^\times$, so by Proposition 3.18 there is a Grothendieck topology $\mathcal{G}$ on $\mathtt{C}$ such that $\mathcal{G}_c = K_S$.

(b) Apply (a) to the property Φ' defined by

$$\Sigma \text{ satisfies } \Phi' \;\Leftrightarrow\; \Sigma^c \text{ does not satisfy } \Phi. \tag{3.56}$$

□

From the above proposition, we immediately deduce the following duality between prime set topologies.

Theorem 3.24 (Duality for prime set topologies)**.** *For $\mathcal{G}$ a prime set topology on $\mathtt{C}$, there is a unique prime set topology $\mathcal{G}^*$ on $\mathtt{C}$ such that:*

$$\begin{aligned} &\{* \xrightarrow{p} *\}_{p\in\Sigma} \text{ is a } \mathcal{G}\text{-covering sieve} \\ \Leftrightarrow\;\; &\{* \xrightarrow{p} *\}_{p\notin\Sigma} \text{ is not a } \mathcal{G}^*\text{-covering sieve.} \end{aligned} \tag{3.57}$$

This is an inclusion-reversing bijection from the poset of prime set topologies to itself. Moreover, a prime set Σ is $\mathcal{G}$-significant if and only if it is $\mathcal{G}^*$-big, and $\mathcal{G}$-big if and only if it is $\mathcal{G}^*$-significant.

Example 3.25.

(a) *Let $\mathcal{G}$ be the trivial Grothendieck topology, with as only covering sieve the maximal one. Then all prime sets Σ are significant. The* dual topology *$\mathcal{G}^*$ is the one where $\{* \xrightarrow{p} *\}_{p\in\Sigma}$ is a covering sieve for all prime sets Σ (even the empty one). So $\mathcal{G}^*$ is the empty Grothendieck topology.*

(b) *Let $\mathcal{G}$ be the atomic Grothendieck topology, with as covering sieves the nonempty sieves. The dual topology $\mathcal{G}^*$ has as covering sieves the sieves containing some p^k for every prime p. Note that $\mathcal{G}$ is of finite type, but $\mathcal{G}^*$ is not.*

(c) *Let $\mathcal{G}$ be the topology with as covering sieves the sieves containing a power of 2. Then a prime set Σ is $\mathcal{G}$-big if and only if $2 \in \Sigma$ if and only if Σ is $\mathcal{G}$-significant. So $\mathcal{G} = \mathcal{G}^*$ (we say $\mathcal{G}$ is self-dual).*

(d) *More generally, let $\mathcal{A}$ be an upwards closed family of prime sets, such that for each Σ, either $\Sigma \in \mathcal{A}$ or $\Sigma^{\mathrm{c}} \in \mathcal{A}$ and not both. Then for $\mathcal{G}$ the corresponding prime set topology, $\mathcal{A}$ is equal to the family of $\mathcal{G}$-big prime sets and to the family of $\mathcal{G}$-significant prime sets. So $\mathcal{G}$ is self-dual.*

(e) *Let $\mathcal{G}$ be the Grothendieck topology with as covering sieves the sieves*

$$\{* \xrightarrow{n_i} *\}_{i\in I} \tag{3.58}$$

such that for each finite set of primes $q_1, \ldots, q_k$ there is some n_i such that $q_j \nmid n_i$ for all $j = 1, \ldots, k$. Then a prime set is $\mathcal{G}$-big if and only if it is infinite, and $\mathcal{G}$-significant if and only if it is cofinite. The covering sieves for the dual topology are the sieves $\{* \xrightarrow{n_i} *\}_{i\in I}$ such that for each infinite sequence of primes $p_1, p_2, p_3, \ldots$ there is an $i \in I$ such that all prime divisors of n_i are in the sequence.

3.4.3 Equivariant sheaves

In Proposition 3.5, we gave an alternative description of the Arithmetic Site in terms of equivariant sheaves. We showed

$$\mathbb{N}^\times_+\text{-}\mathsf{Sets} \;\simeq\; \mathsf{Sh}_{\mathbb{Q}^\times_+}(\mathbb{Q}^\times_+), \tag{3.59}$$

where the right hand side is the topos of equivariant sheaves corresponding to the discrete group $\mathbb{Q}^\times_+$ acting by left multiplication on the space $\mathbb{Q}^\times_+$. The open sets are the subsets $U \subseteq \mathbb{Q}^\times_+$ such that $n \cdot U \subseteq U$ for all $n \in \mathbb{N}^\times_+$.

Note that $\mathbb{Q}^\times_+$ with the latter topology is not sober, so in that sense it does not represent the underlying locale in an optimal way. The sobrification is given by the space $\mathbb{S}_\mathbb{Q}$ defined as follows.

- The elements of $\mathbb{S}_\mathbb{Q}$ are formal infinite products $\prod_{p\text{ prime}} p^{e_p}$ with each $e_p \in \mathbb{Z} \cup \{+\infty\}$ and such that $e_p < 0$ for at most finitely many primes p. The poset structure is given by the division relation (we say $s \mid s'$ if $s' = ns$ for n a supernatural number).
- The topology is the Scott topology: open sets are precisely the upwards closures of families of rational numbers. We write the open sets in the ideal notation (analogous to open sets in $\mathbb{S}$):

$$(q_i)_{i\in I} = \{s \in \mathbb{S}_\mathbb{Q} \;:\; \exists i \in I,\ q_i \mid s\}. \tag{3.60}$$

Proposition 3.26. *There is an equivalence of categories*

$$\mathbb{N}_+^\times\text{-}\mathsf{Sets} \;\simeq\; \mathsf{Sh}_{\mathbb{Q}_+^\times}(\mathbb{S}_\mathbb{Q}) \tag{3.61}$$

Proof. Completely analogous to the proof of Proposition 3.5. The underlying locales of the two topological spaces are the same. □

The following result is probably known, but the author of the thesis did not find any references.

Proposition 3.27. *Let G be a discrete group acting on a sober topological space X. Then the subtoposes of $\mathsf{Sh}_G(X)$ with enough points are in bijective correspondence with the G-invariant sober subspaces $Y \subseteq X$. The subtopos corresponding to $Y \subseteq X$ is $\mathsf{Sh}_G(Y)$.*

Proof. A possible site of definition for $\mathsf{Sh}_G(X)$ is $\mathcal{O}_G(X)$ with

- as objects the open sets $U \subseteq X$;
- as morphisms $U \to V$ the elements $g \in G$ such that $g(U) \subseteq V$;
- as covering sieves the sieves $\{g_i : U_i \to U\}_{i\in I}$ such that

$$\bigcup_{i\in I} g_i(U_i) = U. \tag{3.62}$$

This is the site of definition given by Johnstone in [Joh02a, Example A.2.1.11(c)], see also Subsection 1.3.5. The subtoposes of $\mathsf{Sh}_G(X)$ then correspond to the Grothendieck topologies on the category $\mathcal{O}_G(X)$ that are finer than the topology described above. The slice category of $\mathcal{O}_G(X)$ over X is the category of opens $\mathcal{O}(X)$, and the induced Grothendieck topology is the canonical Grothendieck topology. The corresponding slice topos is $\mathsf{Sh}(X) \simeq \mathsf{Sh}_G(X)/\mathbf{y}X$. Grothendieck topologies on $\mathcal{O}_G(X)$ can be lifted to the slice category $\mathcal{O}(X)$, and if the original Grothendieck topology has enough points, then the induced one has enough points as well (see Subsection 1.3.5). But then the lifted Grothendieck topology corresponds to a sober subspace $Y \subseteq X$, and this completely determines the Grothendieck topology on $\mathcal{O}_G(X)$: the covering sieves are precisely the sieves $\{g_i : U_i \to U\}_{i\in I}$ such that

$$\left(\bigcup_{i\in I} g_i(U_i)\right) \cap Y \;=\; U \cap Y. \tag{3.63}$$

The above equation is equivalent to

$$\left(\bigcup_{i\in I} g(g_i(U_i))\right) \cap g(Y) \;=\; g(U) \cap g(Y), \tag{3.64}$$

but also, by the definition of Grothendieck topology, equivalent to

$$\left(\bigcup_{i\in I} g(g_i(U))\right) \cap Y \;=\; g(U) \cap Y. \tag{3.65}$$

The equivalence of (3.64) and (3.65) shows that Y is G-invariant. Indeed, if $g(Y) \neq Y$, then there is a sieve $\{g_i : U_i \to U\}_{i\in I}$ such that $\left(\bigcup_{i\in I} g_i(U_i)\right) \cap Y = U \cap Y$ but $\left(\bigcup_{i\in I} g_i(U_i)\right) \cap g(Y) \neq U \cap g(Y)$. This is not the case.

Now from the Comparison Lemma it is easy to see that $\mathcal{O}_G(X)$ with the Grothendieck topology we just discussed, gives the same topos as $\mathcal{O}_G(Y)$. So the corresponding subtopos is equivalent to $\mathsf{Sh}_G(Y)$. Conversely, for each G-invariant sober subspace $Y \subseteq X$ we can construct a Grothendieck topology on $\mathcal{O}_G(X)$ such that the category of sheaves is $\mathsf{Sh}_G(Y)$. ☐

Corollary 3.28. *Let $\mathcal{G}$ be a topology with enough points on* $\mathtt{C}$ *and let $S \subseteq \mathbb{S}$ be the sober subspace such that $\mathcal{G}_c = K_S$. Define*

$$S_{\mathbb{Q}} \;=\; \{s \in \mathbb{S}_{\mathbb{Q}} \;:\; \exists n \in \mathbb{N}_+^\times,\; ns \in S\}. \tag{3.66}$$

Then there is an equivalence of categories

$$\mathsf{Sh}(\mathtt{C}, \mathcal{G}) \;\simeq\; \mathsf{Sh}_{\mathbb{Q}_+^\times}(S_{\mathbb{Q}}). \tag{3.67}$$

Example 3.29. *(a) Consider the atomic topology $\mathcal{G}_{\mathrm{at}}$ on* $\mathtt{C}$. *Then*

$$\mathsf{Sh}(\mathtt{C}, \mathcal{G}_{\mathrm{at}}) \;\simeq\; \mathsf{Sh}_{\mathbb{Q}_+^\times}(*) \;\simeq\; \mathbb{Q}_+^\times\text{-}\mathsf{Sets}. \tag{3.68}$$

(b) Consider the gcd-topology *$\mathcal{G}_{\mathrm{gcd}}$ with covering sieves the sieves $\{* \xrightarrow{n_i} *\}_{i \in I}$ with* $\gcd(n_i : i \in I) = 1$. *Then*

$$\mathsf{Sh}(\mathtt{C}, \mathcal{G}_{\mathrm{gcd}}) \;\simeq\; \mathsf{Sh}_{\mathbb{Q}_+^\times}(Z) \tag{3.69}$$

with $Z = \{\alpha \cdot \prod_{q \neq p} q^\infty \;:\; \alpha \in \mathbb{Q}_+^\times,\; p \text{ prime}\} \cup \{\prod_p p^\infty\}$. Note that $\operatorname{Spec}(\mathbb{Z})$ *is homeomorphic to the quotient space $Z/\mathbb{Q}_+^\times$. So we can see $\mathsf{Sh}_{\mathbb{Q}_+^\times}(Z)$ as a "noncommutative" version of* $\operatorname{Spec}(\mathbb{Z})$. *The automorphism group of the point $\prod_{q \neq p} q^\infty$ for $\mathsf{Sh}_{\mathbb{Q}_+^\times}(Z)$ is $\{\alpha \in \mathbb{Q}_+^\times : v_p(\alpha) = 0\}$.*

3.4.4 Notes about finiteness conditions

With the criterion discussed below Definition 2.8, it is easy to see that $\mathbb{S}$ is a spectral space. So $\mathsf{PSh}(\mathtt{D})$ is a coherent topos and the coherent subtoposes are in bijection with the patches $S \subseteq \mathbb{S}$. Note that by "coherent subtopos" we mean a subtopos that is coherent and such that the embedding geometric morphism is coherent.

An obvious question is if the analogous results hold for the Arithmetic Site. It turns out that this is not the case, because $\mathsf{PSh}(\mathtt{C})$ is itself not coherent.

Proposition 3.30. *The Arithmetic Site* $\mathsf{PSh}(\mathtt{C})$ *is not a coherent topos.*

Proof. Using the result from Beke [Bek04, Theorem 3.3] we see that $\mathsf{PSh}(\mathtt{C})$ is a coherent topos if and only if $\mathtt{C}$ has finite fc-colimits, see [Bek04, Definition 2.1]. This means that for any diagram in $\mathtt{C}$ there are a finite number of cocones through which every other cone factors. Let I be an index category with two objects and no morphisms, and let $D : I \to \mathtt{C}$ be the functor sending both objects of I to $*$. There is a cocone $* \xrightarrow{1} * \xleftarrow{n} *$ on D, for every $n \in \mathbb{N}_+^\times$. These cocones are all minimal: they do not factor through another different cocone. So it is impossible to find a finite list of cocones through which every other cocone factors. ☐

Remark 3.31. *The above proof works as well if* C *is given by an arbitrary infinite monoid. On the other hand, if* $\mathsf{C} = M^{\mathrm{op}}$ *for* M *a finite monoid, then* $\mathsf{PSh}(\mathsf{C})$ *is coherent. In fact,* C *can be an arbitrary finite category here, see SGA 4 [sga72, Exposé VI, Exercice 2.17(g)].*

Remark 3.32. *In SGA 4 [sga72, Exercice 2.16(c)], a topological space* X *with an action of a discrete group* G *is considered. It is shown that if the topos* $\mathsf{Sh}_G(X)$ *is quasi-separated, then in particular* G *is finite or* X *is empty. Note that we proved* $\mathsf{PSh}(\mathsf{C}) \simeq \mathsf{Sh}_{\mathbb{Q}_+^\times}(\mathbb{Q}_+^\times)$ *(see Proposition 3.5 for statement and notations). This gives an alternative proof that* $\mathsf{PSh}(\mathsf{C})$ *is not a quasi-separated topos (so in particular, not a coherent topos).*

The statement of SGA 4 [sga72, Exercice 2.16(c)] suggests that for studying spaces with an action of a discrete group, we need different methods for the case where the discrete group is infinite. This is a well-known heuristic in noncommutative geometry.

Chapter 4

An arithmetic topos for integer matrices

In the previous chapter we studied the topos $\mathbb{N}_+^\times$-Sets, which appears in the work of Connes–Consani [CC14] as the underlying topos of their Arithmetic Site. We gave alternative proofs for the classification of the points by the double quotient

$$\widehat{\mathbb{Z}}^\times \backslash \mathbb{A}_f / \mathbb{Q}^\times. \tag{4.1}$$

These proofs are more easily generalized to the topos M-Sets for a different monoid M, as we will demonstrate in this chapter.

Here we consider the topos $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$, with

$$\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}) = \{a \in \mathrm{M}_2(\mathbb{Z}) : \det(a) \neq 0\}, \tag{4.2}$$

as a monoid under multiplication. We will show that the points of this topos are classified up to isomorphism by the double quotient

$$\mathrm{GL}_2(\widehat{\mathbb{Z}}) \backslash \mathrm{M}_2(\mathbb{A}_f) / \mathrm{GL}_2(\mathbb{Q}) \tag{4.3}$$

(note the similarity to the case of the Arithmetic Site). It turns out that this double quotient also classifies the abelian groups $\mathbb{Z}^2 \subseteq A \subseteq \mathbb{Q}^2$ up to isomorphism. This gives an alternative to the similar classification of these groups up to isomorphism by Mal'cev in [Mal38]. In Section 4.5, we study to what extent the double quotient (4.3) lends itself to calculations. We provide an alternative proof for the isomorphism $\mathrm{Ext}^1(\mathbb{Q}, \mathbb{Z}) \cong \mathbb{A}_f/\mathbb{Q}$ and then give an adelic criterion for when two extensions are isomorphic (as abelian groups).

Note that $\mathbb{N}_+^\times$ is the free commutative monoid with the prime numbers as generators. In particular, the prime numbers are indistinguishable from each other: for each permutations of the prime numbers, there is an induced automorphism of $\mathbb{N}_+^\times$. This in turn induces a topos automorphism of $\mathbb{N}_+^\times$-Sets. An important implication is that the topos $\mathbb{N}_+^\times$-Sets contains no information at all about the Riemann Hypothesis. This is one of the reasons why the tropical semiring (as structure sheaf) is so important in the approach of Connes and Consani. In Subsection 4.3 we compute the topos automorphisms of $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$. This topos is much more rigid — in particular, each automorphism acts trivially on the space of points. So in some sense, the topos $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$ contains more arithmetic information than $\mathbb{N}_+^\times$-Sets; and maybe the information in $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-Sets can (partially) replace the role played by the structure sheaf in Connes–Consani [CC14], leading to an even more "algebraic" approach.

An alternative take on this can be found in Subsection 4.4, where the monoid $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$ is studied. It is the submonoid of $\bar{\mathrm{P}}(\mathbb{Z})$ consisting of the matrices with nonzero determinant, where

$$\bar{\mathrm{P}}(R) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \ : \ a, b \in R \right\}. \tag{4.4}$$

for R a commutative ring. These sets of matrices are used by Connes and Consani in [CC18] to study parabolic $\mathbb{Q}$-lattices. We will show that the topos points of $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$-Sets agree with the points of the Arithmetic Site (if we do not take into account the structure sheaf). Moreover, the zeta function naturally associated to $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$ is the Riemann zeta function $\zeta(s)$, and the group of topos automorphisms is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

In Section 4.2.3, we discuss the relationship of $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-Sets with Conway's big picture (as introduced in Conway [Con96]). We consider an embedding of the big picture $\mathfrak{P}$ in the quotient

$$\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}) \tag{4.5}$$

(this embedding already appeared in Plazas [Pla13]). We give an explicit formula for the hyper-distance on $\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$, extending the hyper-distance on the big picture. Then we show that the zeta function associated to the big picture is

$$\zeta_{\mathfrak{P}}(s) = \frac{\zeta(s)\zeta(s-1)}{\zeta(2s)} \tag{4.6}$$

with $\zeta(s)$ the Riemann zeta function. Note that the Riemann zeta function associated to $\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ is $\zeta(s)\zeta(s-1)$. The latter is a special case of a result from Saito [Sai14], but it is also implicit in the work of Connes and Marcolli [CM06], who showed that $\zeta(s)\zeta(s-1)$ is the partition function for their GL_2-system. Note that $\zeta(s)\zeta(s-1)$ is the Hasse–Weil zeta function for $\mathbb{P}^1_{\mathbb{Z}}$. This hints at an interpretation of $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-Sets in terms of algebraic geometry.

Another link with the work of Connes and Marcolli [CM06] is that their GL_2-system $\mathcal{A}$ can be interpreted as an algebra of operators on a vector space internal to the topos $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-Sets. In more down-to-earth terms: there is some vector space E equipped with a linear left $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-action, such that the GL_2-system acts faithfully on E in an equivariant way. Indeed, let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and consider

$$E = \bigoplus_y \ell^2(\Gamma\backslash G_y) \tag{4.7}$$

with $y = (\rho, \tau) \in \mathrm{M}_2(\widehat{\mathbb{Z}}) \times \mathfrak{H}$, for $\mathfrak{H}$ the upper half-plane, and

$$G_y = \{ g \in \mathrm{GL}_2^+(\mathbb{Q}) \ : \ g\rho \in \mathrm{M}_2(\widehat{\mathbb{Z}}) \}$$

and let $a \in \mathcal{A}$ act as $\pi_y(a)$ on $\ell^2(\Gamma\backslash G_y)$, where π_y is the usual representation as constructed by Connes and Marcolli [CM06, Proposition 1.23]. Then $\mathcal{A}$ acts in an equivariant way, provided we equip E with the following left $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-action: if $\xi \in \ell^2(\Gamma\backslash G_y)$, then $a \cdot \xi \in \ell^2(\Gamma\backslash G_{y'})$ with $y' = (a \cdot \rho, a \cdot \tau)$ and moreover

$$(a \cdot \xi)(g) = \xi(ga) \tag{4.8}$$

for all $g \in G_{y'}$. Note that a can have negative determinant, but this issue is resolved by considering the identification

$$\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{+}(\mathbb{Z}). \tag{4.9}$$

4.1 The topos of $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-sets

Let $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}) = \{a \in \mathrm{M}_2(\mathbb{Z}) : \det(a) \neq 0\}$ be the regular integral 2×2-matrices, considered as a monoid under multiplication. In order to study the topos $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$, we will follow a strategy very similar to the strategy in the previous chapter.

We observe that $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$ is equivalent to the category of presheaves on $\mathtt{M}^{\mathrm{op}}$, where $\mathtt{M}$ is the monoid $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$, seen as a category with a unique object $*$. There is an equivalence

$$\mathsf{PSh}(\mathtt{M}^{\mathrm{op}})/\mathbf{y}(*) \;\simeq\; \mathsf{PSh}(\mathtt{M}^{\mathrm{op}}/*). \tag{4.10}$$

Note that, up to equivalence, $\mathtt{M}^{\mathrm{op}}/*$ is the opposite of the poset $\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$. Here the partial order on $\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ is defined as

$$a \leq b \quad \Leftrightarrow \quad \exists m \in \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}),\ b = ma. \tag{4.11}$$

By the discussion in Chapter 2, we find

$$\mathsf{PSh}(\mathtt{M}^{\mathrm{op}}/*) \;\simeq\; \mathsf{Sh}(X) \tag{4.12}$$

where $X = \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ with as open sets the upwards closed sets. Note that the space X is not sober. Its sobrification is given by the dcpo of filters on the poset $(\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}))^{\mathrm{op}}$, equipped with the Scott topology (see Chapter 2). In the next section we will show that the dcpo of filters can be written as $\mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}})$, using the relation with torsion-free abelian groups of rank 2.

The relation with rank 2 torsion-free abelian groups of rank 2 already appears in the following theorem.

Proposition 4.1. *The category of points for the topos* $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$ *is equivalent to the category with*

- *as objects the torsion-free abelian groups of rank* 2*;*
- *as morphisms the injective morphisms of abelian groups.*

Proof. Analogous to how we proved the corresponding statement for the monoid $\mathbb{N}_+^\times$, in Section 3.2. □

Proposition 4.2. *The topos* $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$ *is equivalent to* $\mathsf{Sh}_{\mathrm{GL}_2(\mathbb{Q})}(X)$ *where*

$$X = \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{GL}_2(\mathbb{Q})$$

with $U \subseteq X$ *open if and only if it is closed under left multiplication by elements of* $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$. *The (discrete) group* $\mathrm{GL}_2(\mathbb{Q})$ *acts by right multiplication.*

Proof. Analogous to the proof of Proposition 3.5, after identifying the right $\mathrm{GL}_2(\mathbb{Q})$-action with a left action of $\mathrm{GL}_2(\mathbb{Q})^{\mathrm{op}}$. □

4.2 Description of the slice topos

In Theorem 4.5 we will prove the double quotient formula

$$\mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\mathbb{A}_f)/\,\mathrm{GL}_2(\mathbb{Q}) \tag{4.13}$$

for the points of $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$. In order to arrive at this formula, we will study the points of the slice topos

$$\mathsf{Sh}(\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})) \longrightarrow \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})\text{-}\mathsf{Sets}. \tag{4.14}$$

Afterwards, we show that $\mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\mathbb{A}_f)$ and $\mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}})$ have explicit combinatorial descriptions (the action of $\mathrm{GL}_2(\mathbb{Q})$ on the right is however very complicated).

4.2.1 The poset of abelian groups $\mathbb{Z}^2 \subseteq M \subseteq \mathbb{Q}^2$

Because the topological space $X = \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ from the previous section is not sober, it does not describe all topos-theoretical points of $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$. In this section we compute the sobrification $\hat{X}$ using the theory of torsion-free abelian groups of rank 2. Recall from Chapter 2 that the sobrification can be computed as the dcpo of filters on $(\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}))^{\mathrm{op}}$, with the Scott topology; or equivalently, as the ind-category of $\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$, with the Scott topology.

The trick will be to interpret the poset $\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ as the poset of finitely generated abelian groups M with $\mathbb{Z}^2 \subseteq M \subseteq \mathbb{Q}^2$. Here the abelian group associated to $a \in \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ is

$$M_a = \{m \in \mathbb{Q}^2 : am \in \mathbb{Z}^2\}. \tag{4.15}$$

Clearly, $M_a = M_b$ if and only if $a = ub$ for $u \in \mathrm{GL}_2(\mathbb{Z})$. To see that the map $a \mapsto M_a$ is surjective, take an arbitrary M and look at the left $\mathrm{M}_2(\mathbb{Z})$-ideal $I = \{a \in \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}) : am \in M,\ \forall m \in M\}$. This ideal is a principal ideal by Newman–Pierce [NP69] and it contains a nonzero natural number N, so it must be generated by some $a \in \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$.

So the dcpo of filters on $(\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}))^{\mathrm{op}}$ is equivalently the ind-completion of the poset of finitely generated abelian groups M with $\mathbb{Z}^2 \subseteq M \subseteq \mathbb{Q}^2$. It is easy to see that the ind-completion is the poset of all (not necessarily finitely generated) abelian groups M with $\mathbb{Z}^2 \subseteq M \subseteq \mathbb{Q}^2$.

We claim that the latter poset is isomorphic to $\mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}})$, where

$$\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} \tag{4.16}$$

is the ring of *profinite integers*, and the poset structure is given by

$$a \leq b \quad \Leftrightarrow \quad \exists m \in \mathrm{M}_2(\widehat{\mathbb{Z}}),\ b = ma. \tag{4.17}$$

Proposition 4.3. *There is a natural isomorphism of posets*

$$\mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}}) \;\cong\; \{\textit{abelian groups } M \textit{ such that } \mathbb{Z}^2 \subseteq M \subseteq \mathbb{Q}^2\},$$

where the partial ordering is given on the left by the division relation (4.17) and on the right by the inclusion relation.

Proof. There is a natural isomorphism of posets

$$\mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}}) = \prod_p \mathrm{GL}_2(\mathbb{Z}_p)\backslash \mathrm{M}_2(\mathbb{Z}_p) \tag{4.18}$$

where the product is over all prime numbers. Similarly, the poset of abelian groups $\mathbb{Z}^2 \subseteq M \subseteq \mathbb{Q}^2$ can be written as a product $\prod_p \hat{X}_p$ with

$$\hat{X}_p = \{\mathbb{Z}_p\text{-modules } M \text{ such that } \mathbb{Z}_p^2 \subseteq M \subseteq \mathbb{Q}_p^2\}, \tag{4.19}$$

combine for example Fuchs [Fuc15, Chapter 8, Lemma 5.1] and Roggenkamp–Huber-Dyson[RHD70, Theorem 9.14]. The bijection is given by sending a family $\{M(p)\}_p$ to $\bigcap_p (M(p) \cap \mathbb{Q}^2)$, and conversely sending M to the family $\{M \otimes \mathbb{Z}_p\}_p$.

It remains to show that $\hat{X}_p = \mathrm{GL}_2(\mathbb{Z}_p)\backslash \mathrm{M}_2(\mathbb{Z}_p)$. So take an arbitrary $\mathbb{Z}_p$-module M with $\mathbb{Z}_p^2 \subseteq M \subseteq \mathbb{Q}_p^2$. By Krylov–Tuganbaev [KT08, Corollary 11.8], there are only four indecomposable $\mathbb{Z}_p$-modules up to isomorphism: $\mathbb{Z}_p/p^n\mathbb{Z}_p$, $\mathbb{Q}_p/\mathbb{Z}_p$, $\mathbb{Z}_p$ and $\mathbb{Q}_p$. In our case, this means M is isomorphic to $\mathbb{Z}_p^2$ or $\mathbb{Z}_p \oplus \mathbb{Q}_p$ or $\mathbb{Q}_p^2$, and consequently $\mathbb{Q}_p^2/M \cong (\mathbb{Q}_p/\mathbb{Z}_p)^k$ for some $k \in \{0,1,2\}$. So we can write M as the kernel of a map $f : \mathbb{Q}_p^2 \to (\mathbb{Q}_p/\mathbb{Z}_p)^2$ (the projection on $\mathbb{Q}_p^2/M$ followed by the direct summand inclusion $\mathbb{Q}_p^2/M \hookrightarrow (\mathbb{Q}_p/\mathbb{Z}_p)^2$). In the long exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p^2, \mathbb{Z}_p^2) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p^2, \mathbb{Q}_p^2) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p^2, (\mathbb{Q}_p/\mathbb{Z}_p)^2) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p^2, \mathbb{Z}_p^2) \longrightarrow \dots$$

we have $\mathrm{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p^2, \mathbb{Z}_p^2) = 0$, again by Krylov–Tuganbaev [KT08, Corollary 11.8]. So $f : \mathbb{Q}_p^2 \to (\mathbb{Q}_p/\mathbb{Z}_p)^2$ can be lifted to a map $g : \mathbb{Q}_p^2 \to \mathbb{Q}_p^2$ and

$$M = \{m \in \mathbb{Q}_p^2 : g(m) \in \mathbb{Z}_p^2\}. \tag{4.20}$$

The map g is given by left multiplication by an element $a \in \mathrm{M}_2(\mathbb{Q}_p)$, and from $\mathbb{Z}_p^2 \subseteq M$ we deduce that in fact $a \in \mathrm{M}_2(\mathbb{Z}_p)$. We write

$$M = M_a = \{m \in \mathbb{Q}_p^2 : am \in \mathbb{Z}_p^2\}. \tag{4.21}$$

Clearly, if $u \in \mathrm{GL}_2(\mathbb{Z}_p)$ then $M_a = M_{ua}$. Conversely, take $a, b \in \mathrm{M}_2(\mathbb{Z}_p)$ such that $M_a = M_b = M$. Consider the maps $\pi \circ a$ and $\pi \circ b$ with $\pi : \mathbb{Q}_p^2 \to (\mathbb{Q}_p/\mathbb{Z}_p)^2$ the quotient map. Both maps can be written as the quotient map $\mathbb{Q}_p^2 \to \mathbb{Q}_p^2/M$ followed by an injection $i : \mathbb{Q}_p^2/M \hookrightarrow (\mathbb{Q}_p/\mathbb{Z}_p)^2$. Because $\mathbb{Q}_p^2/M$ is in each case embedded as a direct summand (by definition), two different embeddings $i, i' : \mathbb{Q}_p^2/M \hookrightarrow (\mathbb{Q}_p/\mathbb{Z}_p)^2$ are conjugated. Endomorphisms of $(\mathbb{Q}_p/\mathbb{Z}_p)^2$ can be identified with elements of $\mathrm{M}_2(\mathbb{Z}_p)$, so automorphisms correspond to elements of $\mathrm{GL}_2(\mathbb{Z}_p)$. This shows $b = ua$ with $u \in \mathrm{GL}_2(\mathbb{Z}_p)$. □

Corollary 4.4. *Let* $X = \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ *as above.*

- *The dcpo of filters on* X^{op} *is* $\hat{X} = \mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}})$ *with the partial order given by the division relation (4.17).*
- *The sobrification of* X *(as a topological space) is* $\hat{X} = \mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}})$ *with the Scott topology.*

Recall that the open sets for the Scott topology on $\hat{X}$ can be written as

$$(a_i)_{i\in I} \;=\; \{x : \exists i \in I,\ a_i \leq x\} \tag{4.22}$$

where $(a_i)_{i\in I}$ is a family of elements in $X = \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$.

Because $\hat{X}$ is the sobrification of X, we know that $\mathsf{Sh}(X) \simeq \mathsf{Sh}(\hat{X})$, and the space of points for $\mathsf{Sh}(X)$ is precisely $\hat{X}$. The geometric morphism

$$\mathsf{Sh}(X) \longrightarrow \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})\text{-}\mathsf{Sets} \tag{4.23}$$

induces a map on topos-theoretic points, and it is easy to see that this is exactly the map

$$a \mapsto M_a = \{m \in \mathbb{Q}^2 : am \in \widehat{\mathbb{Z}}^2\}. \tag{4.24}$$

This map is surjective, because any rank 2 torsion-free abelian group is isomorphic to an abelian group M with $\mathbb{Z}^2 \subseteq M \subseteq \mathbb{Q}^2$. Two elements $a, b \in \hat{X}$ determine isomorphic points of $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$ if and only if $M_a \cong M_b$. We observe that any isomorphism $M_a \to M_b$ extends to an isomorphism $g \in \mathrm{GL}_2(\mathbb{Q})$, which shows that $a = bg$ up to left multiplication by some $u \in \mathrm{GL}_2(\widehat{\mathbb{Z}})$.

The results from the section are summarized in the following theorem. As in the case for $\mathbb{N}_+^\times$, we use the finite adeles instead of the profinite integers to write down the double quotient, because the right $\mathrm{GL}_2(\mathbb{Q})$-action on $\mathrm{M}_2(\widehat{\mathbb{Z}})$ is only partially defined. We call $a, b \in \mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}})$ equivalent if there is an $g \in \mathrm{GL}_2(\mathbb{Q})$ such that $a = bg$. Then the equivalence classes are in bijective correspondence with the elements of $\mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\mathbb{A}_f)/\mathrm{GL}_2(\mathbb{Q})$.

Theorem 4.5. *The set of isomorphism classes of topos-theoretic points of* $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$ *can be written as a double quotient*

$$\mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\mathbb{A}_f)/\mathrm{GL}_2(\mathbb{Q}) \tag{4.25}$$

with $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$ *the ring of* finite adeles*.*

Remark 4.6. *The idea of using finite adeles to describe subgroups* $\mathbb{Z}^2 \subseteq A \subseteq \mathbb{Q}^2$ *up to isomorphism is not new. The standard approach seems to be the one introduced by Mal'cev [Mal38] in 1938, see Fuchs [Fuc73, Theorem 93.4]. However, the description from the above theorem is a bit different and was not found by the author of this thesis in previous works.*

Note that both the original description by Mal'cev and the variation above are rather unpractical. For two matrices $a, b \in \mathrm{M}_2(\widehat{\mathbb{Z}})$ *it is in general very difficult to determine if they represent the same element of* $\mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\mathbb{A}_f)/\mathrm{GL}_2(\mathbb{Q})$*.*

Because $\hat{X}$ is the sobrification of X, their associated locales are the same. So we can state the following variant of Proposition 4.2.

Proposition 4.7. *The topos* $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$ *is equivalent to* $\mathsf{Sh}_{\mathrm{GL}_2(\mathbb{Q})}(\hat{X})$ *where*

$$\hat{X} = \mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}})$$

with the Scott topology. The (discrete) group $\mathrm{GL}_2(\mathbb{Q})$ *acts by right multiplication.*

Proof. This follows directly from Proposition 4.2 because the locales associated to $\hat{X}$ and X are the same. □

4.2.2 Combinatorial description

In this section we give a concrete description of the posets $X = \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ and $\hat{X} = \mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}})$. Recall that the poset relation was given by

$$a \leq b \quad \Leftrightarrow \quad \exists m \in M,\ a = mb, \tag{4.26}$$

with $M = \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ in the first case and $M = \mathrm{M}_2(\widehat{\mathbb{Z}})$ in the second case.

Note that

$$\mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}}) \;=\; \prod \mathrm{GL}_2(\mathbb{Z}_p)\backslash \mathrm{M}_2(\mathbb{Z}_p) \tag{4.27}$$

so we can see an element $a \in \mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}})$ as a family

$$(a_p)_p \;\text{ with } a_p \in \mathrm{GL}_2(\mathbb{Z}_p)\backslash \mathrm{M}_2(\mathbb{Z}_p) \text{ for all primes } p. \tag{4.28}$$

Moreover with this notation we see that $a \leq b$ for $a, b \in \mathrm{M}_2(\widehat{\mathbb{Z}})$ if and only if $a_p \leq b_p$ for all primes p. So we will fix one prime p and look at $\mathrm{GL}_2(\mathbb{Z}_p)\backslash \mathrm{M}_2(\mathbb{Z}_p)$ in detail.

Because $\mathbb{Z}_p$ is a principal ideal domain, every element of $\mathrm{M}_2(\mathbb{Z}_p)$ can be brought into its *Hermite normal form*, by multiplying on the left with elements of $\mathrm{GL}_2(\mathbb{Z}_p)$, see MacDuffee [Mac33, Theorem 22.1]. This Hermite normal form is a matrix

$$\begin{pmatrix} p^k & z \\ 0 & p^l \end{pmatrix} \tag{4.29}$$

with $z = z_0 + z_1 p + z_2 p^2 + \dots$ satisfies $z_i = 0$ for $i \geq l$; but we allow both $k = \infty$ and $l = \infty$ with the convention that $p^\infty = 0$ (in the case $l = \infty$ there is no restriction on z). This Hermite normal form is unique whenever the determinant is nonzero (so k, l both finite), see MacDuffee [Mac33, Theorem 22.2]. In this case, we easily find

$$\begin{pmatrix} p^k & z \\ 0 & p^l \end{pmatrix} \leq \begin{pmatrix} p^r & z' \\ 0 & p^s \end{pmatrix} \tag{4.30}$$

if and only if $k \leq r$, $l \leq s$ and $z' \equiv p^{r-k} z \mod p^s$.

Let $a, b \in \mathrm{GL}_2(\mathbb{Z}_p)\backslash \mathrm{M}_2(\mathbb{Z}_p)$ with nonzero determinant. Then we say that a and b are *adjacent* if

- $a \leq b$ and $\det(b) = p\det(a)$ (in this case we write $a \to b$);
- $b \leq a$ and $\det(a) = p\det(b)$ (in this case we write $b \to a$).

In this way, we can interpret $\mathrm{GL}_2(\mathbb{Z}_p)\backslash \mathrm{M}_2(\mathbb{Z}_p)$ as a (directed) graph. Some additional definitions: let $a \in \mathrm{GL}_2(\mathbb{Z}_p)\backslash \mathrm{M}_2(\mathbb{Z}_p)$, then

- we define the *level* $\lambda(a)$ as the largest integer such that $p^{\lambda(a)} \mid a$;
- we define the *niveau* $\nu(a)$ as $\nu(a) = v_p(\det a) - 2\lambda(a)$.

If we multiply a matrix by a scalar n, then the level increases by $v_p(n)$ and the niveau stays the same. So we could alternatively define the niveau of a as the valuation of the determinant of $\frac{1}{N}a$, with N the greatest common divisor for the entries of a.

For $a \to b$, we easily compute that either $\lambda(b) = \lambda(a)$ and $\nu(b) = \nu(a) + 1$, or $\lambda(b) = \lambda(a) + 1$ and $\nu(b) = \nu(a) - 1$. Moreover, the latter can only occur for at most one b. In Table 4.1 we give a complete description of the directed graph structure. Figure 4.1 illustrates the situation for $p = 2$. The elements with $\lambda \leq 2$ and $\nu \leq 4$ are drawn with an edge between each two adjacent elements.

	$\#\left\{\begin{matrix} a \to b \\ \lambda(b)=\lambda(a)\end{matrix}\right\}$	$\#\left\{\begin{matrix} a \to b \\ \lambda(b)>\lambda(a)\end{matrix}\right\}$	$\#\left\{\begin{matrix} b \to a \\ \lambda(b)=\lambda(a)\end{matrix}\right\}$	$\#\left\{\begin{matrix} b \to a \\ \lambda(b)<\lambda(a)\end{matrix}\right\}$
$\lambda(a)=0$ $\nu(a)=0$	$p+1$	0	0	0
$\lambda(a)=0$ $\nu(a)>0$	p	1	1	0
$\lambda(a)>0$ $\nu(a)=0$	$p+1$	0	0	$p+1$
$\lambda(a)>0$ $\nu(a)>0$	p	1	1	p

Table 4.1: Four types in $\mathrm{GL}(\mathbb{Z}_p)\backslash\,\mathrm{M}_2(\mathbb{Z}_p)$ and their adjacent elements.

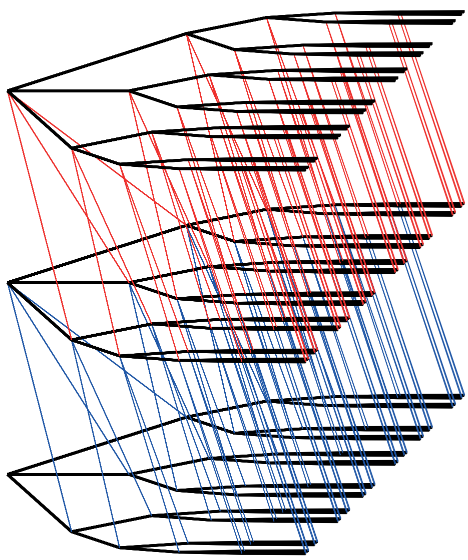


Figure 4.1: A (truncated) picture of $\mathrm{GL}_2(\mathbb{Z}_2)\backslash\,\mathrm{M}_2(\mathbb{Z}_2)$.

For an element of $\mathrm{GL}_2(\mathbb{Z}_p)\backslash\,\mathrm{M}_2(\mathbb{Z}_p)$ with zero determinant, the following matrices are unique representatives:

$$\begin{pmatrix} p^k & z \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & p^l \end{pmatrix} \tag{4.31}$$

for some $k \in \{0,1,2,3,\dots\}$ and $z \in \mathbb{Z}_p$, or $l \in \{0,1,2,\dots\} \cup \{\infty\}$. The poset structure on the latter matrices can be summarized as

$$\begin{pmatrix} p^k & z \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} p^{k+r} & p^r z \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & p^l \end{pmatrix} \leq \begin{pmatrix} 0 & 0 \\ 0 & p^{l+s} \end{pmatrix} \leq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We define the level $\lambda(a)$ again as the largest integer such that $p^{\lambda(a)} \mid a$, or $\lambda(a) = \infty$ for a the zero matrix. For any matrix with zero determinant we write $\nu(a) = \infty$.

It is easy to see from the explicit representatives above that there is a bijection between the nonzero elements with zero determinant and the "paths" in $\mathrm{GL}_2(\mathbb{Z}_p)\backslash\,\mathrm{M}_2(\mathbb{Z}_p)$, where by "path" we mean a subset $\{a_n\}_{n\in\mathbb{N}}$ with $\lambda(a_n) =$

$\lambda(a_0)$, $\nu(a_n) = n$ and $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$. The path associated to b with $\det(b) = 0$ is explicitly given by

$$\{a \leq b \ : \ \lambda(a) = \lambda(b)\}. \tag{4.32}$$

This finishes our description of the poset $\mathrm{GL}_2(\mathbb{Z}_p)\backslash \mathrm{M}_2(\mathbb{Z}_p)$. Note that the determinant map

$$\mathrm{GL}_2(\mathbb{Z}_p)\backslash \mathrm{M}_2(\mathbb{Z}_p) \longrightarrow \mathbb{Z}_p^\times\backslash\mathbb{Z}_p \tag{4.33}$$

is also easy to visualize when keeping Figure 4.1 in mind. Moreover, we can identify $\mathbb{Z}_p^\times\backslash\mathbb{Z}_p$ with $\mathbb{N}\cup\{\infty\}$ using the map

$$\xi : \mathbb{N}\cup\{\infty\} \longrightarrow \mathbb{Z}_p^\times\backslash\mathbb{Z}_p, \qquad n \mapsto p^n \tag{4.34}$$

with the convention $p^\infty = 0$. Note that $\xi(n+m) = \xi(n)\xi(m)$. Taking the product over all primes p gives a description of $\mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}})$ and the corresponding determinant map

$$\mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}}) \longrightarrow \widehat{\mathbb{Z}}^\times\backslash\widehat{\mathbb{Z}}.$$

Here $\widehat{\mathbb{Z}}^\times\backslash\widehat{\mathbb{Z}}$ can be identified with the Steinitz numbers or supernatural numbers

$$\mathbb{S} = \prod_p \mathbb{N}\cup\{\infty\}. \tag{4.35}$$

The supernatural numbers already made their appearance in Chapter 3.

4.2.3 Relation to Conway's big picture

Conway's *big picture* (introduced in Conway [Con96]) is the graph with vertex set $\mathbb{Q}_+ \times \mathbb{Q}/\mathbb{Z}$ and edges defined by a hyper-distance δ.

We use the notations from Le Bruyn [LB18]: the big picture is denoted by $\mathfrak{P}$, and for each $X = (M, \frac{g}{h}) \in \mathbb{Q}_+ \times \mathbb{Q}/\mathbb{Z}$ we consider the matrices

$$\alpha_X = \begin{pmatrix} M & \frac{g}{h} \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{GL}_2^+(\mathbb{Q}) \tag{4.36}$$

where $\mathrm{SL}_2(\mathbb{Z})$ acts by left multiplication on $\mathrm{GL}_2^+(\mathbb{Q})$, the subgroup of $\mathrm{GL}_2(\mathbb{Q})$ consisting of the matrices with positive determinant. The *hyper-distance* δ is then given by

$$\delta(X,Y) = \det(\alpha_{XY}\alpha_X\alpha_Y^{-1}) \tag{4.37}$$

with α_{XY} the smallest strictly positive rational number such that $\alpha_{XY}\alpha_X\alpha_Y^{-1} \in \mathrm{M}_2(\mathbb{Z})$; further, there is an edge between X and Y whenever $\delta(X,Y)$ is a prime number (see Le Bruyn [LB18, p. 7] for all this).

We claim that we can embed $\mathfrak{P}$ as a full subgraph of $\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$. The embedding is given on vertices by

$$\mathbb{Q}_+ \times \mathbb{Q}/\mathbb{Z}, \quad (M, \tfrac{g}{h}) = X \mapsto \begin{pmatrix} MN & \frac{g}{h}N \\ 0 & N \end{pmatrix} = \beta_X \tag{4.38}$$

where $N \in \mathbb{N}_+$ is minimal such that $\frac{g}{h}N \in \mathbb{Z}$. We can assume $0 \leq \frac{g}{h} < 1$ and then β_X is in Hermite normal form so the mapping is injective. Further, the greatest common divisor of the entries of β_X is 1 (i.e. the entries are coprime) and conversely every $a \in \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ with this property can be written as $a = \beta_X$ for some $X \in \mathbb{Q}_+ \times \mathbb{Q}/\mathbb{Z}$. In the notations of Subsection 4.2.2.

$$\mathfrak{P} \;=\; \{a \in \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}) \;:\; \lambda_p(a) = 0 \text{ for all primes } p\} \tag{4.39}$$

where $\lambda_p(a)$ is the level of a at prime p.

In the following proposition, we will see that the poset structure on $\mathfrak{P}$ as a subset of $\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ is the same as the poset structure on the big picture as introduced in Le Bruyn [LB18, Definition 1], i.e. the one given by

$$X \leq Y \text{ iff } \delta(1,Y) = \delta(X,Y)\delta(1,X). \tag{4.40}$$

A fortiori, the edges of the underlying graphs are the same.

Proposition 4.8. *Consider the* hyper-distance $\tilde{\delta}$ *on* $\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ *given by*

$$\tilde{\delta}(a,b) = \det(a')\det(b')$$

where $x = a \wedge b$ and $a = a'x$, $b = b'x$. Then:

- *(a)* $\log \tilde{\delta}(a,b)$ *is the* weighted distance *between a and b where an edge $x \to y$ with* $\det(y) = p\det(x)$ *has weight* $\log(p)$*;*
- *(b)* $\tilde{\delta}(x,y) = \delta(x,y)$ *for $x, y \in \mathfrak{P}$;*
- *(c) the poset structure on $\mathfrak{P}$ as a subset of* $\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ *agrees with the poset structure from (4.40) above by Le Bruyn [LB18, Definition 1];*
- *(d) $\mathfrak{P}$ is a fundamental domain for the monoid action of $\mathbb{N}_+^\times$ on*

  $$\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$$

  by scalar multiplication.

Proof.

(a) We denote the weighted distance between a and b by $d(a,b)$. Note that

$$d(a,b) = \sum_p d_p(a_p, b_p)$$

where a_p and b_p are the projections on $\mathrm{GL}_2(\mathbb{Z}_p)\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}_p)$ of a resp. b, and d_p is the weighted distance function in $\mathrm{GL}_2(\mathbb{Z}_p)\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}_p)$, where every edge has weight $\log(p)$. As a weighted graph, $\mathrm{GL}_2(\mathbb{Z}_p)\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}_p)$ can be identified with

$$X_p = \{a \in \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}) \;:\; a_q = 1 \text{ for all } q \neq p\}.$$

It follows from the above that it is enough to prove the statement for $a, b \in X_p$. First assume that $x = a \wedge b = 1$, $a \neq 1$ and $b \neq 1$. We claim that then $a, b \in \mathfrak{P}$, i.e. $p \nmid a$ and $p \nmid b$. Indeed, suppose $p \mid a$. Any divisor $y \leq b$ with $\det(y) = p$ then also divides a. This shows $b = 1$, a contradiction. So $p \nmid a$ and analogously $p \nmid b$. Take a path of minimal length from a to b. We can assume that this path does not leave $\mathfrak{P}$, so it is of length $\det(a)\det(b)$. Now suppose that $x = a \wedge b \neq 1$. Again we take a path of minimal length from a to b and we can assume that this path does not leave

$$\uparrow x = \{a \in X_p \;:\; a \geq x\}.$$

Multiplication by x^{-1} on the right is an isometry from $\uparrow x$ to X_p, and replaces a by a', b by b' and x by 1. From the previous case we find

$$d(a,b) = d(a',b') = \det(a')\det(b').$$

(b) This follows directly from (1).
(c) It is enough to show that

$$a \leq b \text{ iff } \tilde{\delta}(1,b) = \tilde{\delta}(a,b)\tilde{\delta}(1,a).$$

This easily follows from (1), for example by induction on the number of prime divisors of $\tilde{\delta}(x,y)$ (counted with multiplicity).
(d) This is clear from the description of $\mathfrak{P}$ as consisting of the matrices for which the entries are coprime.

□

From now on, we use the notation $\mathfrak{M} = \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$. We can associate to it the *zeta function*

$$\zeta_{\mathfrak{M}}(s) = \sum_{x\in\mathfrak{M}} \det(x)^{-s}. \tag{4.41}$$

It is proved by Saito in [Sai14] that

$$\zeta_{\mathfrak{M}}(s) = \zeta(s)\zeta(s-1). \tag{4.42}$$

Note that this is the same as the Hasse–Weil zeta function for $\mathbb{P}^1_{\mathbb{Z}}$.

Remark 4.9. *Saito in [Sai14] considers* $\mathrm{GL}_n(R)\backslash \mathrm{M}_n^{\mathrm{ns}}(R)$ *for n a natural number and R a principal ideal domain, and shows that the zeta function is equal to*

$$\zeta_R(s)\zeta_R(s-1)\cdots\zeta_R(s-n+1). \tag{4.43}$$

(Saito uses the notation $P_{\mathrm{M}(n,R)^\times,\deg}(\exp(-s))$ *for this zeta function.) In our case, it follows directly from the Hermite normal form that the number of elements in $\mathfrak{M}$ with determinant n is given by $\sigma(n)$, so*

$$\zeta_{\mathfrak{M}}(s) = \sum_n \sigma(n)n^{-s} = \zeta(s)\zeta(s-1). \tag{4.44}$$

So following Saito's approach is not necessary in this easy case. Also, the zeta function $\zeta(s)\zeta(s-1)$ for $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ *already appeared implicitly in the work of Connes and Marcolli (see e.g. [CM06]), when they show that it is the partition function of their* GL_2*-system.*

Proposition 4.10. *Consider the big picture $\mathfrak{P}$ as a subgraph of $\mathfrak{M}$. Then its zeta function is given by*

$$\zeta_{\mathfrak{P}}(s) \;=\; \sum_{x\in\mathfrak{P}} \det(x)^{-s} \;=\; \frac{\zeta(s)\zeta(s-1)}{\zeta(2s)}.$$

Proof. In Proposition 4.8 we proved that $\mathfrak{P}$ is a fundamental domain for the action of $\mathbb{N}_+^\times$ on $\mathfrak{M}$. So $\mathfrak{M}$ can be written as a disjoint union

$$\mathfrak{M} \;=\; \bigsqcup_{n\in\mathbb{N}_+} n\cdot\mathfrak{P} \tag{4.45}$$

The zeta function for $n\cdot\mathfrak{P}$ is given by $n^{-2s}\cdot\zeta_{\mathfrak{P}}(s)$, so we get

$$\zeta_{\mathfrak{M}}(s) = \sum_{n\in\mathbb{N}_+} n^{-2s}\cdot\zeta_{\mathfrak{P}}(s) = \zeta(2s)\zeta_{\mathfrak{P}}(s). \tag{4.46}$$

The statement then follows from $\zeta_{\mathfrak{M}}(s) = \zeta(s)\zeta(s-1)$. □

Remark 4.11. *We can also write the zeta function for* $\mathfrak{P}$ *as*

$$\zeta_{\mathfrak{P}}(s) = \sum_{X \in \mathbb{Q}_+ \times \mathbb{Q}/\mathbb{Z}} \det(\beta_X)^{-s}. \tag{4.47}$$

with β_X *as in (4.38). The analogous definition*

$$\xi_{\mathfrak{P}}(s) = \sum_{X \in \mathbb{Q}_+ \times \mathbb{Q}/\mathbb{Z}} \det(\alpha_X)^{-s} \tag{4.48}$$

with α_X *as in (4.36) is not well-defined, because for a fixed* n *there are infinitely many* $X \in \mathbb{Q}_+ \times \mathbb{Q}/\mathbb{Z}$ *with* $\det(\alpha_X) = n$.

4.3 Automorphisms of $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-Sets

The goal of this subsection is to show that the group of automorphisms of $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-Sets (up to natural isomorphism) is rather small: it is isomorphic to the group of characters $\mathrm{Hom}(\mathbb{Q}^\times, \{1, -1\})$. Intuitively, it is not surprising that it is a small group, when taking into account the result of Stephenson [Ste69] that all monoid automorphisms of $\mathrm{M}_2(\mathbb{Z})$ are inner. But Stephenson's proof uses the idempotents in $\mathrm{M}_2(\mathbb{Z})$; if we consider the submonoid $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$, then there are more automorphisms, as shown in the following proposition.

Proposition 4.12. *Let* $\vartheta : \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}) \to \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ *be an automorphism of monoids. Then we can find* $g \in \mathrm{GL}_2(\mathbb{Z})$ *and a character* $\chi : \mathbb{Q}^\times \to \{1, -1\}$ *such that*

$$\vartheta(a) = \chi(\det(a))\, gag^{-1}.$$

In particular, ϑ *preserves the determinant. Conversely, any* ϑ *as above determines an automorphism of* $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$.

Proof. The groupification of $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ is $\mathrm{GL}_2(\mathbb{Q})$, so ϑ is the restriction of a group automorphism $\hat{\vartheta} : \mathrm{GL}_2(\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q})$. This $\hat{\vartheta}$ can be written as

$$\hat{\vartheta}(a) = \hat{\chi}(a)\, gag^{-1} \tag{4.49}$$

with $\hat{\chi} : \mathrm{GL}_2(\mathbb{Q}) \to \mathbb{Q}^\times$ a character and $g \in \mathrm{GL}_2(\mathbb{Q})$ (use the result by Hua [Die51, Supplement, Theorem 1] and keep in mind that $a \mapsto \det(a)\,(a^t)^{-1}$ is conjugation by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so this is an inner automorphism). Note that $\hat{\chi}$ is necessarily trivial on the commutator subgroup $\mathrm{SL}_2(\mathbb{Q})$ of $\mathrm{GL}_2(\mathbb{Q})$, so we can write $\hat{\chi}(a) = \chi(\det(a))$ for some character $\chi : \mathbb{Q}^\times \to \mathbb{Q}^\times$.

Further, ϑ induces a poset automorphism

$$\bar{\vartheta} \;:\; \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}) \longrightarrow \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}) \tag{4.50}$$

In particular, it preserves the elements that have exactly one strictly smaller element. These correspond to the elements $a \in \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ with $\det(a) = \pm p$ for some prime number p. So if $\det(a) = \pm p$ then up to sign $\det(\vartheta(a)) = q$ with q prime. Moreover

$$\det(\vartheta(a)) = \chi(\det(a))^2 \det(a) \tag{4.51}$$

so in fact $\det(\vartheta(a)) = \pm p$ (with the same sign) and $\chi(\det(a))^2 = 1$. Every $a \in \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ has a *Smith normal form* $a = udv$ where u, v are units and d is a diagonal matrix. Using the Smith normal form, we see that $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ is generated by units and matrices of prime determinant. As a corollary $\det(\vartheta(a)) = \det(a)$ and $\chi(\det(a))^2 = 1$ for any $a \in \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$. We still need to deduce that $g \in \mathrm{GL}_2(\mathbb{Z})$ from $\hat{\vartheta}$ sending $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ to itself. Note that $gag^{-1} \in \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ for all $a \in \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$. Further, for any $b \in \mathrm{M}_2(\mathbb{Z})$ we can find $\lambda \in \mathbb{Z}$ big enough such that $b + \lambda I_2$ is in $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$. This implies that $gbg^{-1} \in \mathrm{M}_2(\mathbb{Z})$ and in this way conjugation by g defines a ring automorphism of $\mathrm{M}_2(\mathbb{Z})$. This shows $g \in \mathrm{GL}_2(\mathbb{Z})$. □

We want to relate the monoid automorphisms of $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ to the topos automorphisms of $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-Sets. This is something that can be done for more general monoids M, so we will prove the more general results if possible.

A *cancellative monoid* is a monoid such that $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow b = c$. In a cancellative monoid M being left invertible is equivalent to being right invertible. The group of (left and right) invertible elements will be denoted by $M^\times$. As in the case of groups there is a (group) morphism

$$\begin{aligned} M^\times &\longrightarrow \mathrm{Aut}(M) \\ g &\mapsto \iota_g \end{aligned}$$

with $\iota_g(m) = gmg^{-1}$. The image is the normal subgroup of inner automorphisms, denoted by $\mathrm{Inn}(M) \subseteq \mathrm{Aut}(M)$. We write

$$\mathrm{Out}(M) = \frac{\mathrm{Aut}(M)}{\mathrm{Inn}(M)}$$

for the group of outer automorphisms. With these notations we can formulate an immediate corollary to Proposition 4.12.

Corollary 4.13. *There is a group isomorphism*

$$\mathrm{Out}(\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})) \cong \mathrm{Hom}(\mathbb{Q}^\times, \{1, -1\}).$$

Note that every $f \in \mathrm{Hom}(\mathbb{Q}^\times, \{1, -1\})$ is uniquely determined by choosing a value for $f(-1)$ and for $f(p)$ for all primes p.

We now want to determine the topos automorphisms of $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-Sets. We first present a criterion describing which topos automorphisms of M-Sets (with M an arbitrary monoid) arise from monoid automorphisms of M.

Proposition 4.14. *Let M be an arbitrary monoid and let $\Theta : M$-Sets $\to M$-Sets be an equivalence such that $G\Theta \simeq G$ for $G : M$-Sets $\to$ Sets the forgetful functor. Then $\Theta \simeq \vartheta^*$ for some monoid automorphism $\vartheta : M \to M$. Conversely, $G\vartheta^* \simeq G$ for every monoid automorphism $\vartheta : M \to M$.*

Proof. For ϑ an automorphism of M, the equivalence ϑ^* satisfies $G\vartheta^* = G$ and can be reconstructed from the monoid map

$$\begin{aligned} \mathrm{Nat}(G, G) &\to \mathrm{Nat}(G, G) \\ \alpha_m &\mapsto \alpha_m \vartheta^* = \alpha_{\vartheta(m)}. \end{aligned}$$

Here α_m, $m \in M$ is the natural transformation given by $\alpha_m(n) = m \cdot n$ for all M-sets N and $n \in N$ (every natural transformation $G \Rightarrow G$ is of this form). We denote horizontal composition for natural transformations by juxtaposition and we denote vertical composition with $\circ$. Note that any monoid map $\mathrm{Nat}(G,G) \to \mathrm{Nat}(G,G)$ is of the form $\alpha_m \mapsto \alpha_{\vartheta(m)}$ for some ϑ.

Now take some Θ as in the proposition and take an equivalence $\varphi : G\Theta \Rightarrow G$. Then we can construct a monoid map

$$\begin{aligned} \mathrm{Nat}(G,G) &\to \mathrm{Nat}(G,G) \\ \alpha_m &\mapsto \varphi \circ \alpha_m\Theta \circ \varphi^{-1}. \end{aligned}$$

We can find a monoid automorphism $\vartheta : M \to M$ such that

$$\alpha_{\vartheta(m)} = \varphi \circ \alpha_m\Theta \circ \varphi^{-1}, \tag{4.52}$$

in other words, such that the diagram

$$\begin{array}{ccc} \Theta N & \xrightarrow{\varphi} & N \\ {\scriptstyle\alpha_m}\big\downarrow & & \big\downarrow{\scriptstyle\alpha_{\vartheta(m)}} \\ \Theta N' & \xrightarrow[\varphi]{} & N' \end{array} \tag{4.53}$$

commutes for any M-sets N and N' and any $m \in M$. But this means that φ is equivariant, so it defines a natural isomorphism $\varphi : \Theta \Rightarrow \vartheta^*$. □

Corollary 4.15. *Let M be a cancellative monoid. Then the topos automorphisms Θ of M-*Sets *up to natural isomorphism satisfying $G\Theta \simeq G$ form a group isomorphic to* $\mathrm{Out}(M)$.

Proof. We already showed that every autoequivalence Θ with $G\Theta \simeq G$ is induced by an automorphism of M (up to natural isomorphism). We still need to determine when $\vartheta^* \simeq \zeta^*$ for ϑ, ζ two automorphisms of M. If $\zeta = \iota_g \circ \vartheta$ for some $g \in M^\times$, then left multiplication by g defines a natural isomorphism $\vartheta \Rightarrow \zeta$. Conversely, if $\varphi : \vartheta \Rightarrow \zeta$ is a natural isomorphism, then by functoriality $\varphi(ma) = \varphi(m)a$ for all $m, a \in M$. So $\varphi(m) = gm$ for some $g \in M^\times$. Because φ is equivariant, we see that $\zeta = \iota_g \circ \vartheta$. In conclusion, ϑ and ζ are naturally isomorphic if and only if they differ by an inner automorphism. □

We now return to the case $M = \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$. We first show that the criterion from Corollary 4.15 is satisfied for all topos automorphisms.

Lemma 4.16. *Let*

$$\Theta : \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})\text{-}\mathsf{Sets} \longrightarrow \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})\text{-}\mathsf{Sets}$$

be an autoequivalence. Then $G\Theta \simeq G$.

Proof. Note that $G \simeq p^*$ with p the point corresponding to the identity matrix. Then $G\Theta$ has a right adjoint and preserves finite limits so $G\Theta \simeq q^*$ for some point q. For N an $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-set, we can write

$$G\Theta N \simeq q^*N = \varinjlim_{m \leq x} N \tag{4.54}$$

for some $x \in \mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}})$. We claim that $q^* \simeq p^* \simeq G$. To show this, we take $N = \{0, 1\}$ with the $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-action

$$m \cdot x = \begin{cases} x & \text{if } \det(m) = 1 \\ 0 & \text{if } \det(m) \neq 1 \end{cases}. \tag{4.55}$$

We compute

$$\varinjlim_{m \leq x} N = \begin{cases} N & \text{if } x \in X \\ 1 & \text{if } x \in \hat{X} \setminus X \end{cases} \tag{4.56}$$

(recall the notations $X = \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ and $\hat{X} = \mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}})$). Because Θ is an equivalence, it cannot send N to 1 (if so, Θ^{-1} would not preserve the terminal object). So $x \in X$ and because X is precisely the orbit of the identity matrix under the (partial) $\mathrm{GL}_2(\mathbb{Q})$-action, we see that $G\Theta \simeq q^* \simeq p^* \simeq G$. □

We now combine Corollary 4.15 with Lemma 4.16. In order to determine the topos automorphisms of $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-Sets we just need to compute $\mathrm{Out}(\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z}))$. By Corollary 4.13, we can identify the outer automorphisms with the characters $\mathbb{Q}^\times \to \{1, -1\}$. This leads us to the following description.

Theorem 4.17. *The group of topos automorphisms of* $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-Sets *up to natural isomorphism can be identified with the group of characters*

$$\chi : \mathbb{Q}^\times \to \{1, -1\}$$

under pointwise multiplication. In particular, the topos automorphisms act trivially on the space of points.

4.4 Alternative: the $ax + b$ monoid

In a recent paper of Connes and Consani [CC18], the subsets

$$\bar{\mathrm{P}}(R) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \ : \ a \in R, \ b \in R \right\} \tag{4.57}$$

are introduced, for an arbitrary commutative ring R. They are used to study *parabolic* $\mathbb{Q}$-lattices, see [CC18, Definition 6.1, p. 50]. For example, the parabolic $\mathbb{Q}$-lattices up to commensurability are given by

$$\mathcal{C}_{\mathbb{Q}}^0 = \mathrm{P}^+(\mathbb{Q})\backslash \left(\bar{\mathrm{P}}(\mathbb{A}_f) \times \mathrm{P}^+(\mathbb{R}) \right), \tag{4.58}$$

where the superscript + means that we take the subset of matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a > 0$, see [CC18, Theorem 6.1.(ii), p. 52].

In this section we will study the submonoid $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z}) \subset \bar{\mathrm{P}}(\mathbb{Z})$ consisting of matrices with nonzero determinant, and the associated topos $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$-Sets. We will show that this topos has three nice properties as a setting for the Riemann Hypothesis:

(a) its topos points are given by $\widehat{\mathbb{Z}}^\times \backslash \mathbb{A}_f / \mathbb{Q}^\times$, so by Le Bruyn [LB16] the topos points are the same as for the underlying topos of the Arithmetic Site of Connes–Consani [CC14];

(b) its associated zeta function is the Riemann zeta function $\zeta(s)$;
(c) its group of topos automorphisms is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and acts trivially on the space of points.

So $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$-Sets resembles the Arithmetic Site (without structure sheaf), but this time there are only two automorphisms.

The units of the monoid $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$ are given by the matrices $\begin{pmatrix} \pm 1 & b \\ 0 & 1 \end{pmatrix}$, and it is easy to see that the elements of the quotient $Y = \bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})^{\times}\backslash\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$ have unique representatives of the form $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $a > 0$. There is an obvious inclusion $Y \subset X$, with $X = \mathrm{GL}_2(\mathbb{Z})\backslash\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ as before. This inclusion puts a partial ordering on Y, and this is the same partial ordering as the one arising naturally from the division relation on $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$. On both posets, we consider the topology with upwards closed sets as opens. Then the inclusion is continuous, and the topology on Y is the subspace topology with respect to $Y \subset X$.

Proposition 4.18. *Consider $Y \subset X$ as above.*
(a) The sobrification $\hat{Y}$ of Y corresponds to the subset of elements of $\hat{X}$ that have a representative of the form

$$\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \tag{4.59}$$

for $z \in \widehat{\mathbb{Z}}$.
(b) Under the isomorphism of posets

$$\begin{aligned} \hat{X} &\cong \{\text{abelian groups } M \text{ such that } \mathbb{Z}^2 \subseteq M \subseteq \mathbb{Q}^2\} \\ a &\mapsto M_a \end{aligned}$$

the elements of $\hat{Y}$ correspond to the abelian groups M with $M \subseteq \mathbb{Q} \oplus \mathbb{Z}$.

Proof. (a) $\hat{Y}$ is the closure of Y under the strong topology. This is easy to compute using the combinatorial description from Subsection 4.2.2.
(b) Straightforward computation.

□

Proposition 4.19. *The category of points for the topos $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$-Sets is equivalent to the category with*

- *as objects the torsion-free abelian groups M of rank 2, equipped with a surjective morphism $\pi_M : M \to \mathbb{Z}$;*
- *as morphisms $f : M \to N$ the injective morphisms such that $\pi_N \circ f = \pi_M$.*

Proof. The category of points is equivalent to the ind-category on $\bar{\mathrm{P}}^{\mathrm{ns}}$, interpreted as a category with one object. We will embed $\bar{\mathrm{P}}^{\mathrm{ns}}$ as a full subcategory of the category $\mathcal{L}$, where $\mathcal{L}$ has as objects the abelian groups M equipped with a surjective morphism $\pi_M : M \to \mathbb{Z}$, and as maps the injective morphisms $f : M \to N$ with $\pi_N \circ f = \pi_M$. We do this by sending the unique object $*$ of $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$ to the abelian group $\mathbb{Z}^2$ with the surjection $\pi_2 : \mathbb{Z}^2 \to \mathbb{Z},\ (x, y) \mapsto y$. The matrix $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ is sent to the endomorphism f with $f(x, y) = (ax + by, y)$. Note that $\pi_2 \circ f = \pi_2$ and conversely every f with this property comes from a matrix in $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$. To every formal filtered colimit $(\varinjlim_i *)$ in $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$ we can associate the object $\varinjlim_i \mathbb{Z}^2$ (as abelian groups). As projection $\varinjlim_i \mathbb{Z}^2 \to \mathbb{Z}$ we take the map induced by the projections on the second factor $\pi_2 : \mathbb{Z}^2 \to \mathbb{Z}$. Now take an arbitrary torsion-free abelian group M of rank 2, with a surjection $\pi_M : M \to \mathbb{Z}$. We know that M is a filtered colimit of its free rank 2 submodules $F_i \cong \mathbb{Z}^2$, and we can restrict to the F_i such that $\pi_M|_{F_i}$ is surjective (this is a cofinal system).

In order to show that the morphisms in $\mathcal{L}$ agree with the morphisms in the ind-category of $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$, we need that $\mathcal{L}(\varinjlim_i \mathbb{Z}^2, \varinjlim_j \mathbb{Z}^2) \simeq \varprojlim_i \varinjlim_j \mathcal{L}(\mathbb{Z}^2, \mathbb{Z}^2)$. This follows from the case where $\mathcal{L}$ is replaced by the category of abelian groups. $\square$

There is a commutative diagram of geometric morphisms

$$\begin{array}{ccc} \mathsf{Sh}(Y) & \longrightarrow & \mathsf{Sh}(X) \\ \downarrow & & \downarrow \\ \bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})\text{-}\mathsf{Sets} & \longrightarrow & \bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})\text{-}\mathsf{Sets} \end{array} \tag{4.60}$$

and this induces a commutative diagram between the spaces of points

$$\begin{array}{ccc} \hat{Y} & \longrightarrow & \hat{X} \\ \downarrow & & \downarrow \\ \mathsf{Pts}(\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})\text{-}\mathsf{Sets}) & \longrightarrow & \mathsf{Pts}(\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})\text{-}\mathsf{Sets}) \end{array} . \tag{4.61}$$

The horizontal maps are the obvious maps. The vertical maps are given by $a \mapsto M_a$.

Suppose that $M_a \cong M_b$ in $\mathsf{Pts}(\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})\text{-}\mathsf{Sets})$. We can take representatives

$$a = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} z' & 0 \\ 0 & 1 \end{pmatrix} \tag{4.62}$$

and then $M_a \cong M_b$ implies $z' = qz$ for some $q \in \mathbb{Q}^\times$.

This leads to the following theorem. Note that we have to replace $\widehat{\mathbb{Z}}$ by the finite adeles (again), in order to get an actual $\mathbb{Q}^\times$-action instead of a partial one.

Theorem 4.20. *The topos points of* $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$ *are given by*

$$\widehat{\mathbb{Z}}^\times \backslash \mathbb{A}_f / \mathbb{Q}^\times.$$

In particular, they agree with the topos points of $\mathbb{N}_+^\times$-Sets*, the underlying topos for the Arithmetic Site of Connes–Consani [CC14].*

The embedding $Y \subset X$ allows us to associate a zeta function to $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$, namely

$$\zeta_{\bar{\mathrm{P}}}(s) = \sum_{a \in Y} \det(a)^{-s}. \tag{4.63}$$

Clearly $\zeta_{\bar{\mathrm{P}}}(s)$ is the Riemann zeta function $\zeta(s)$.

Now we show that $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$-$\mathsf{Sets}$ has only two topos automorphisms. We first determine the outer automorphisms of $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$.

Proposition 4.21. *The outer automorphism group of* $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$ *is isomorphic to* $\mathbb{Z}/2\mathbb{Z}$.

Proof. The nonzero integers will be considered as a submonoid, where $n \in \mathbb{Z}$, $n \neq 0$, corresponds to the matrix $\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$. The nonzero integers generate $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$, together with the matrix $t \in \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and its inverse. Every element has a unique representation in the form $t^b a$, with $a, b \in \mathbb{Z}$, $a \neq 0$. Moreover, $nt = t^n n$. It is now easy to see that, if α is an automorphism of $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$, then $\alpha(n) = t^{k(n)}n$, for some $k(n) \in \mathbb{Z}$ depending on n. Because $\alpha(n)$ and $\alpha(2)$ commute, we find

$$k(n) = (n-1)k(2). \tag{4.64}$$

This implies $\alpha(n) = t^{-k(2)} n t^{k(2)}$, so up to an inner automorphism we can assume that $\alpha(n) = n$ for all $n \in \mathbb{Z}$, $n \neq 0$. It is now clear that either

$$\alpha(t^k n) = t^k n \tag{4.65}$$

or

$$\alpha(t^k n) = t^{-k} n. \tag{4.66}$$

□

Corollary 4.22. *The group of topos automorphisms of* $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$-Sets *(up to natural isomorphism) is isomorphic to* $\mathbb{Z}/2\mathbb{Z}$.

Proof. By Corollary 4.15, we need to show that any topos automorphism Θ of $\bar{\mathrm{P}}^{\mathrm{ns}}(\mathbb{Z})$-Sets satisfies $G\Theta \simeq G$, where G is the forgetful functor. The proof of this fact is completely analogous to the proof of Lemma 4.16. □

4.5 Applications

In Theorem 4.5, we proved that the set of points for $\mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$-Sets is given by

$$\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2(\mathbb{A}_f)/\mathrm{GL}_2(\mathbb{Q}) \tag{4.67}$$

and that, in particular, this double quotient classifies abelian groups $\mathbb{Z}^2 \subseteq A \subseteq \mathbb{Q}^2$ up to isomorphism. Here $\mathbb{A}_f = (\prod_p \mathbb{Z}_p) \otimes \mathbb{Q}$ denotes the finite adèles.

In this section we discuss some applications, with as underlying goal to determine to what extent this description is suitable for calculations.

4.5.1 Relation to $\mathrm{Ext}^1(\mathbb{Q}, \mathbb{Z})$

The Ext-group $\mathrm{Ext}^1(\mathbb{Q}, \mathbb{Z})$ can be written as

$$\mathrm{Ext}^1(\mathbb{Q}, \mathbb{Z}) \cong \mathbb{A}_f/\mathbb{Q}. \tag{4.68}$$

For a proof using the long exact sequence we refer to Boardman's note [Boa10]. We also refer to Morava [Mor13] (and the blogpost by Le Bruyn [LB14]) where an analogon for the full ring of adeles is discussed. In this subsection we provide an alternative proof of (4.68) using Theorem 4.5. From this approach we automatically get a criterion describing when two extension of $\mathbb{Q}$ by $\mathbb{Z}$ are isomorphic as abelian groups (equivalent extensions are always isomorphic as abelian groups, but the converse does not hold).

We saw in the proof of Proposition 4.3 that for every subgroup $\mathbb{Z}^2 \subseteq A \subseteq \mathbb{Q}^2$ there is an $x \in \mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}})$ such that

$$A_x = \varinjlim_{m \leq x} \mathbb{Z}^2 \tag{4.69}$$

where the filtered colimit is over the $m \leq x$ with $m \in \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{M}_2^{\mathrm{ns}}(\mathbb{Z})$ and where a transition map $\mathbb{Z}^2 \to \mathbb{Z}^2$ corresponding to $m \leq m'$ is given by a, with a the matrix such that $m' = am$ (we assume that m, m' are in Hermite normal form, in order to fix a matrix representative). Alternatively,

$$A_x = \{(u,v) \in \mathbb{Q}^2 \ : \ x \cdot (u,v) \in \widehat{\mathbb{Z}}^2\} \tag{4.70}$$

where (u,v) is seen as a column vector in $\mathbb{A}_f^2$ on which x acts by matrix multiplication.

We will focus on the subgroups A_x such that the sequence

$$\begin{array}{c} 0 \longrightarrow \mathbb{Z} \longrightarrow A_x \longrightarrow \mathbb{Q} \longrightarrow 0 \\ n \longmapsto (n,0) \\ (u,v) \longmapsto v \end{array} \tag{4.71}$$

is exact. In other words, we consider the subgroups $\mathbb{Z}^2 \subseteq A_x \subseteq \mathbb{Q}^2$ with the properties
(E1) $(u,0) \in A_x$ implies that $u \in \mathbb{Z}$; and
(E2) for all $v \in \mathbb{Q}$ there is an $u \in \mathbb{Q}$ with $(u,v) \in A_x$.
By definition A_x then determines an element $[A_x] \in \mathrm{Ext}^1(\mathbb{Q},\mathbb{Z})$ and it is easy to see that, conversely, every element of $\mathrm{Ext}^1(\mathbb{Q},\mathbb{Z})$ is the class of some A_x.

We now describe the elements $x \in \mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}})$ such that A_x satisfies (E1) and (E2). First we introduce the *supernatural numbers as subset of the profinite integers.*

Definition 4.23. *A* supernatural number *(or Steinitz number) is a profinite integer $s \in \widehat{\mathbb{Z}}$ such that for each prime p its projection on $\mathbb{Z}_p$ (i.e. the pth component) is either 0 or a power of p. The supernatural numbers will be denoted by $\mathbb{S}$. They are a set of representatives for $\widehat{\mathbb{Z}}^\times\backslash\widehat{\mathbb{Z}}$.*

With p^∞ we denote the supernatural number such that the pth component is 0 and such that the qth component is 1 for each $q \neq p$.

For each natural number $n \in \mathbb{N}$ we define $s(n)$ to be the supernatural number such that for each prime p its projection on $\mathbb{Z}_p$ is p^k, where p^k is the largest pth power dividing n.

Note that $s(0) = 0 = \prod_p p^\infty$ and $s(1) = 1$, but these are the only $n \in \mathbb{N}$ for which $s(n) = n$. Our definition of the supernatural numbers agrees with the usual definition, apart from the fact that the supernatural numbers are usually defined abstractly as a monoid under multiplication (not as a subset of the profinite integers).

The supernatural numbers $\mathbb{S}$ come into the picture when considering the Hermite normal form for $x \in \mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}})$. It is given by

$$x = \begin{pmatrix} s & z \\ 0 & s' \end{pmatrix} \tag{4.72}$$

with $s, s' \in \mathbb{S}$ and $z \in \widehat{\mathbb{Z}}$. Note that two matrices in Hermite normal form might describe the same element of $\mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\widehat{\mathbb{Z}})$, for example

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.73}$$

So what are the matrices

$$x = \begin{pmatrix} s & z \\ 0 & s' \end{pmatrix}$$

such that A_x satisfies (E1) and (E2)? First we use that $(u, 0) \in A_x$ if and only if

$$\begin{pmatrix} s & z \\ 0 & s' \end{pmatrix} \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} su \\ 0 \end{pmatrix} \in \widehat{\mathbb{Z}}^2, \tag{4.74}$$

which is the case if and only if $su \in \widehat{\mathbb{Z}}$. If $p \mid s$ then $(\frac{1}{p}, 0) \in A_x$, so it follows that A_x satisfies (E1) if and only if $s = 1$. More generally, $(u, v) \in A_x$ if and only if

$$\begin{pmatrix} s & z \\ 0 & s' \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} su + zv \\ s'v \end{pmatrix} \in \widehat{\mathbb{Z}}^2. \tag{4.75}$$

Now it is easy to see that (E1) and (E2) hold if and only if $s = 1$ and $s' = 0$. So the matrices under consideration are of the form

$$x = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix}.$$

As a group under multiplication, they can be identified with the additive group of profinite integers $\widehat{\mathbb{Z}}$. Further, suppose that

$$x = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad x' = \begin{pmatrix} 1 & z' \\ 0 & 0 \end{pmatrix} \tag{4.76}$$

determine an equivalent extension (i.e. the same element in $\mathrm{Ext}^1(\mathbb{Q}, \mathbb{Z})$). Then $A_{x'} = g \cdot A_x$ for some $g \in \mathrm{GL}_2(\mathbb{Q})$ that preserves both the inclusion of $\mathbb{Z}$ and the projection on $\mathbb{Q}$. We write

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then g preserves $\mathbb{Z}$ if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.77}$$

in other words, if and only if $a = 1$ and $c = 0$. Moreover, g preserves the projection on $\mathbb{Q}$ if and only if

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \tag{4.78}$$

in other words, if and only if $c = 0$ and $d = 1$. So g is of the form

$$g = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

and we get

$$\begin{pmatrix} 1 & z' \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z-b \\ 0 & 0 \end{pmatrix} \tag{4.79}$$

(note that $g \cdot A_x = A_{xg^{-1}}$). This shows

$$\mathrm{Ext}^1(\mathbb{Q},\mathbb{Z}) \;=\; \widehat{\mathbb{Z}}/\mathbb{Z} \;=\; \mathbb{A}_f/\mathbb{Q}. \tag{4.80}$$

Even if A_x and $A_{x'}$ define non-equivalent extensions, it is still possible that they are isomorphic as abelian groups. Any isomorphism $A_x \cong A_{x'}$ is given by conjugation by an element of $\mathrm{GL}_2(\mathbb{Q})$. So if

$$x = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad x' = \begin{pmatrix} 1 & z' \\ 0 & 0 \end{pmatrix}$$

then there is a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$ such that

$$\begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+cz & b+dz \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & z' \\ 0 & 0 \end{pmatrix} \tag{4.81}$$

define the same element in $\mathrm{GL}_2(\widehat{\mathbb{Z}})\backslash \mathrm{M}_2(\mathbb{A}_f)$. This is the case if and only if

$$a + cz \in \widehat{\mathbb{Z}}^\times \quad \text{and} \quad z' = \frac{b+dz}{a+cz}. \tag{4.82}$$

Proposition 4.24. *Consider the partially defined right action of* $\mathrm{PGL}_2(\mathbb{Q})$ *on*

$$\mathrm{Ext}^1(\mathbb{Q},\mathbb{Z}) = \mathbb{A}_f/\mathbb{Q}$$

given by

$$z \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; \frac{b+dz}{a+cz}$$

whenever $a + cz \in \mathbb{A}_f^\times$. *Then two extensions* $A, A' \in \mathrm{Ext}^1(\mathbb{Q},\mathbb{Z})$ *are isomorphic as abelian groups if and only if they are in the same* $\mathrm{PGL}_2(\mathbb{Q})$*-orbit.*

Proof. This follows from the above discussion. Note that for $a + cz \in \mathbb{A}_f^\times$ we can assume that $a + cz \in \widehat{\mathbb{Z}}^\times$, because $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is only defined up to a scalar. □

4.5.2 Relation to Goormaghtigh conjecture

We now try to determine whether some specific extensions in

$$\mathrm{Ext}^1(\mathbb{Q},\mathbb{Z}) = \mathbb{A}_f/\mathbb{Q}$$

are isomorphic as abelian groups. This will reveal some advantages and limitations of the description from Proposition 4.24.

Recall from Definition 4.23 the construction of supernatural numbers as subset of $\widehat{\mathbb{Z}}$, and the specific supernatural numbers $s(n)$ with pth component given by the largest pth power dividing n.

We first consider the set

$$N = \{s(n) : n \in \mathbb{N}\} \;\subseteq\; \mathbb{A}_f/\mathbb{Q} = \mathrm{Ext}^1(\mathbb{Q},\mathbb{Z}); \tag{4.83}$$

when do $s(n)$ and $s(m)$ define isomorphic abelian groups? In other words, when can we find some

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{Q}) \text{ such that } s(m) = \frac{b + d\,s(n)}{a + c\,s(n)} \; ? \tag{4.84}$$

In the following, we use the notation $s(m) \sim s(n)$ when the above holds.

Proposition 4.25.
(a) *if* $s(n) \sim s(0)$ *then* $n \in \{0, 1\}$*;*
(b) *if* $s(n) \sim s(m)$ *for* $n, m \notin \{0, 1\}$*, then* n *and* m *have the same prime divisors;*
(c) $s(p^k) \sim s(p^u)$ *for all primes* p *and integers* $k, u \geq 1$*;*
(d) $s(p^k q^r) \sim s(p^u q^v)$ *for all primes* p, q *and integers* $k, r, u, v \geq 1$*.*

Proof.
(a) Note that $s(0) = 0$, so if $s(n) \sim s(0)$ then $s(n) \in \mathbb{Q}$. It is clear that $n = 0, 1$ are possible. Conversely, if $n \neq 0$ then the pth component of $s(n)$ is 1 for almost all primes p. Together with $s(n) \in \mathbb{Q}$ this shows $s(n) = 1$, so $n = 1$.
(b) Suppose that $s(n) \sim s(m)$, more precisely

$$s(m) = \frac{b + d\,s(n)}{a + c\,s(n)}.$$

Then $a\,s(m) + c\,s(nm) = b + d\,s(n)$. Note that for almost all primes p, the pth components of both $s(n)$ and $s(m)$ are 1. By looking at such a component we see that $a + c = b + d$ and by rescaling we can assume $a + c = 1 = b + d$. Now suppose that there is a prime q such that the qth components of $s(n)$ and $s(m)$ are 1 resp. q^v. Then $q^v = aq^v + cq^v = b + d = 1$.
(c) Note that $s(p^u) = \frac{p^k - p^u}{p^k - 1} + \frac{p^u - 1}{p^k - 1}\, s(p^k)$; this can be checked componentwise.
(d) It is enough to show that there is a solution to the system of equations

$$\begin{cases} b + d = 1 \\ a + c = 1 \\ b + dp^k = ap^u + cp^{k+u} \\ b + dq^r = aq^v + cq^{r+v} \end{cases}.$$

Then $s(p^u q^v) = \frac{b + d\,s(p^k q^r)}{a + c\,s(p^k q^r)}$ and moreover $a + c\,s(p^k q^r) \in \mathbb{A}_f^\times$ (indeed, if the pth component of $a + c\,s(p^k q^r)$ would be zero, then $a + cp^k = 0 = b + dp^k$; together with $a + c = 1 = b + d$ we find $(a, c) = (b, d)$ but this contradicts $b + dq^r = aq^v + cq^{r+v}$). The system of equations has a solution because

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & p^k & p^u & p^{k+u} \\ 1 & q^r & q^v & q^{r+v} \end{vmatrix} = (p^k - 1)(q^r - 1)(p^u - q^v) \neq 0.$$

□

For more general extensions, the situation becomes a lot more complicated. Recall that the *Goormaghtigh conjecture*[1] states that the only natural number solutions to

$$\frac{x^n - 1}{x - 1} = \frac{y^m - 1}{y - 1} \tag{4.85}$$

are $(x, y, n, m) = (2, 5, 5, 3)$ and $(x, y, n, m) = (2, 90, 13, 2)$. The conjecture is still open at the time of writing.

Proposition 4.26 (Relation to Goormaghtigh conjecture)**.** *We have*

$$s(2^4 \cdot 5^2)l^\infty \sim s(2^5 \cdot 5^3)l^\infty$$

for all primes l. Any other solution (p, q, l, k, r), $p \leq q$ of

$$s(p^k q^r)l^\infty \sim s(p^{k+1} q^{r+1})l^\infty$$

gives a counterexample to Goormaghtigh conjecture.

Proof. We can check componentwise that

$$\frac{30\, s(2^4 \cdot 5^2)l^\infty}{31 - s(2^4 \cdot 5^2)l^\infty} = s(2^5 \cdot 5^3). \tag{4.86}$$

Further, if

$$\frac{b + d\, s(p^k q^r)l^\infty}{a + c\, s(p^k q^r)l^\infty} = s(p^{k+1} q^{r+1})l^\infty, \tag{4.87}$$

then we can assume $b + d = 1 = a + c$ like in the proof of (2), and by looking at the components we get

$$\begin{cases} b + d = 1 \\ a + c = 1 \\ b + dp^k = ap^{k+1} + cp^{2k+1} \\ b + dq^r = aq^{r+1} + cq^{2r+1} \\ b = 0 \end{cases}. \tag{4.88}$$

From this we find

$$\begin{cases} 1 = a + c \\ 1 = ap + cp^{k+1} \\ 1 = aq + cq^{k+1} \end{cases} \tag{4.89}$$

so

$$a = -\tfrac{p^{k+1}-1}{p-1}\, c = -\tfrac{q^{r+1}-1}{q-1}\, c \tag{4.90}$$

but this means that $(p, q, k + 1, r + 1)$ is a counterexample to Goormaghtigh conjecture, except when $p^{k+1} = 2^5$ and $q^{r+1} = 5^3$. ☐

[1] Named after the Belgian engineer/mathematician René Goormaghtigh.

Chapter 5

Azumaya toposes

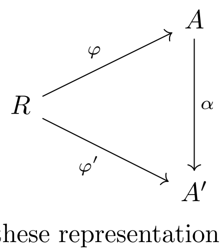
In this chapter, we will show how the study of Grothendieck topologies on the big cell is useful for a problem that is unrelated at first sight: the study of Grothendieck topologies on a category of Azumaya algebras arising naturally in the representation theory of algebras.

Throughout, all algebras R will be associative, unital, finitely generated $\mathbb{C}$-algebras, not necessarily commutative. With $\operatorname{rep}_n(R)$ we denote the affine scheme of all n-dimensional representations of R, that is, all $\mathbb{C}$-algebra maps $R \longrightarrow \mathrm{M}_n(\mathbb{C})$. Conjugation in $\mathrm{M}_n(\mathbb{C})$ defines a PGL_n-action on $\operatorname{rep}_n(R)$, its orbits corresponding to isomorphism classes of n-dimensional representations. By results of Artin [Art69] and Procesi [Pro87] it is known that the geometric points of the quotient scheme $\operatorname{rep}_n(R)/\mathrm{PGL}_n$ classify isomorphism classes of n-dimensional semi-simple representations of R.

In order to classify the isomorphism classes of all n-dimensional representations one has to consider the representation stack of n-dimensional representations $[\operatorname{rep}_n(R)/\mathrm{PGL}_n]$ which by the results of Le Bruyn [LB12] is the functor from the category Comm of all commutative $\mathbb{C}$-algebras to $\mathsf{Groupoids}$ the category of all groupoids

$$[\operatorname{rep}_n(R)/\mathrm{PGL}_n] \ : \ \mathsf{Comm} \longrightarrow \mathsf{Groupoids} \qquad C \mapsto \mathsf{Azu}_n^C(R) \tag{5.1}$$

where the objects of the groupoid $\mathsf{Azu}_n^C(R)$ are the $\mathbb{C}$-algebra maps $\varphi : R \longrightarrow A$ where A is a degree n Azumaya algebra with center C, and morphisms $\alpha : \varphi \longrightarrow \varphi'$ are given by C-algebra morphisms $\alpha : A \longrightarrow A'$ making the diagram below commute.



(5.2)

The information contained in these representation stacks, for varying n, can also be expressed in the following way. Consider the category Azu with objects all Azumaya algebras and with morphisms all $\mathbb{C}$-algebra maps preserving centers. Given an affine $\mathbb{C}$-algebra R we can then consider the covariant functor

$$\mathsf{Alg}(R, -) \ : \ \mathsf{Azu} \longrightarrow \mathsf{Sets} \qquad A \mapsto \mathsf{Alg}_{\mathbb{C}}(R, A) \tag{5.3}$$

and this is a presheaf on the opposite category $\mathsf{Azu}^{\mathrm{op}}$.

We will investigate Grothendieck topologies on $\mathsf{Azu}^{\mathrm{op}}$ for which the functor $\mathsf{Alg}(R,-)$ is a sheaf. In Section 5.1 we study the problem of extending Grothendieck topologies on $\mathsf{Comm}^{\mathrm{op}} = \mathsf{Aff}$ to the category $\mathsf{Azu}^{\mathrm{op}}$. It turns out that this problem is related to Grothendieck topologies on the big cell $\mathtt{D}$. More precisely, for certain couples (J, K), with J a Grothendieck topology on $\mathsf{Comm}^{\mathrm{op}}$ and K a Grothendieck topology of finite type on $\mathtt{D}$, we can construct a combined Grothendieck topology J_K on $\mathsf{Azu}^{\mathrm{op}}$.

In Section 5.2 we will show that the functor $\mathsf{Alg}(R,-)$ on Azu is a sheaf for every Grothendieck topology on $\mathsf{Azu}^{\mathrm{op}}$ coarser than the maximal flat topology, that is the combined Grothendieck topology given by the flat topology on $\mathsf{Comm}^{\mathrm{op}}$ and the atomic topology on $\mathtt{D}$. If we fix an Azumaya algebra A with center C it follows that the functor

$$\mathsf{Comm}_C \longrightarrow \mathsf{Sets} \qquad D \mapsto \mathsf{Alg}_{\mathbb{C}}(R, A \otimes_C D) \tag{5.4}$$

is a sheaf with respect to any Grothendieck topology coarser than the flat topology. The main result of this section shows that this sheaf is in fact representable by an affine scheme over $\mathrm{Spec}(C)$, which we call the Azumaya representation scheme of R associated to the Azumaya algebra A.

In Section 5.3 we look at the topologies J_K that are *trivializing*, i.e. such that every Azumaya algebra is J_K-locally given by matrix algebras. The obvious examples are the topologies J_K with J finer than the étale topology. But we also determine the Grothendieck topologies K such that J_K is trivializing, for J the Zariski topology.

In Section 5.4, we show that the trivializing topologies J_K have enough points whenever J has enough points. More explicitly, we show that the family

$$\mathrm{P}(J, K) = \{\mathrm{M}_s(D) \mid s \in S \text{ and } D \text{ a } J\text{-local commutative algebra}\}$$

is a separating family of points for J_K, where each $\mathrm{M}_s(D)$ is a certain union of matrix algebras over D, similar to the UHF-algebra associated to s.

If J_K is moreover coarser than the maximal flat topology, then $\mathsf{Alg}(R,-)$ is a J_K-sheaf, for R a finitely generated, not necessarily commutative algebra. In this case, we can associate a topos to the algebra R: the slice topos

$$\mathsf{Sh}(\mathsf{Azu}^{\mathrm{op}}, J_K)/\mathsf{Alg}(R,-).$$

For these toposes, we find the family of points

$$\mathrm{P}_R(J, K) = \{R \to \mathrm{M}_s(D) \mid s \in S \text{ and } D \text{ a } J\text{-local commutative algebra}\},$$

which is again separating whenever J has enough points. So the topos-theoretic points we associate to R correspond to certain representations parametrized by J-local commutative algebras.

In Section 5.5 we construct a projective general linear group PGL_s over the complex numbers, for each supernatural number $s \in \mathbb{S}$. The construction is analogous to the construction of UHF-algebras as unions of matrix algebras. Moreover, there is a natural action of PGL_s on the UHF-algebra $\mathrm{M}_s(\mathbb{C})$ and this action satisfies a finiteness condition that will be important later on. The PGL_s-actions satisfying this condition will be called *continuous*.

After studying PGL_s, we take a closer look at the topologies J_K with K the Grothendieck topology on $\mathtt{D}$ induced by the singleton $\{s\}$ (we write $J_K = J_s$). We show that there is an equivalence of categories

$$\mathsf{Sh}(\mathsf{Azu}^{\mathrm{op}}, J_s) \simeq \mathrm{PGL}_s - \mathsf{Sh}(\mathsf{Comm}^{\mathrm{op}}, J).$$

The left hand side is the category of J_s-sheaves on $\mathsf{Azu}^{\mathrm{op}}$. The right hand side is the category of J-sheaves on $\mathsf{Comm}^{\mathrm{op}}$, equipped with a continuous PGL_s-action (the morphisms in the category are PGL_s-equivariant sheaf morphisms). In particular, for n a natural number, we can interpret J_n-sheaves on $\mathsf{Azu}^{\mathrm{op}}$ as sheaves on $\mathsf{Comm}^{\mathrm{op}}$ equipped with a PGL_n-action (every possible action is continuous in this case). Another important case is when we take s to be the maximal supernatural number

$$s = \prod_p p^\infty.$$

Then the Grothendieck topology J_s is the maximal topology as introduced in Hemelaer–Le Bruyn [HLB16]. So sheaves for the maximal topology also have an interpretation in terms of equivariant sheaves on $\mathsf{Comm}^{\mathrm{op}}$.

5.1 Grothendieck topologies on Azumaya algebras

From now on, we take the complex numbers $\mathbb{C}$ as a base field. *Algebras* are associative, not necessarily commutative rings with unit, containing $\mathbb{C}$.

Let C be a commutative algebra. Recall from Demeyer–Ingraham [DI71] that an algebra A is said to be an *Azumaya algebra* over C if and only if

(a) The center $Z(A)$ of A equals C.

(b) There is a *separability idempotent* $e = \sum a_i \otimes b_i \in A \otimes_C A^{\mathrm{op}}$, that is, $\mu(e) = \sum_i a_i b_i = 1$ and $\sum_i x a_i \otimes b_i = \sum_i a_i \otimes b_i x$ for all $x \in A$.

If only the second condition is satisfied we say that A is *separable* over C. Equivalently, A is an Azumaya algebra over C if and only if there is an étale cover

$$\{C \to C_i\}_{i=1}^k$$

such that for each $i \in \{1, \ldots, k\}$ there is an $n_i \in \mathbb{N}_+$ for which $A \otimes_C C_i \cong \mathrm{M}_{n_i}(C_i)$, the algebra of $n_i \times n_i$-matrices with coefficients in C_i. So A is projective over C and we can often assume that A is of constant rank n^2, in which case n will be called the *degree* of A.

Definition 5.1. *With* Azu *we will denote the category having as its objects all finitely generated Azumaya algebras A over commutative algebras, and an algebra morphism $f : A \longrightarrow A'$ is a morphism in* Azu *if it preserves centers, that is if $f(Z(A)) \subset Z(B)$. Note that when A and A' are Azumaya algebras of the same constant degree n this condition is always satisfied.*

Similarly, we will write Comm *for the category of finitely generated commutative algebras.*

We will often invoke the *(Double) Centralizer Theorem* (see Demeyer–Ingraham [DI71, Theorem II.4.3]): let A be an Azumaya algebra with center C and let

$C \subseteq B \subseteq A$ be any subalgebra of A separable over C. Then the *centralizer*

$$A^B = \{a \in A \ : \ \forall b \in B, \ a.b = b.a\} \tag{5.5}$$

is also separable over C and $A^{(A^B)} = B$. If B is in addition an Azumaya algebra over C, then so is A^B and we have

$$A \simeq B \otimes_C A^B \tag{5.6}$$

It is well known that the category Azu_C of all Azumaya algebras with the same center C is a symmetric monoidal category under $\otimes_C$. More generally, if A and B are separable over the commutative ring C, then so is $A \otimes_C B$. An immediate consequence of the double centralizer theorem is:

Proposition 5.2. *If $f_i : A \longrightarrow A_i$ (for $i = 1, 2$) are morphisms in* Azu *then the tensor product*

$$A_1 \otimes_A A_2 \tag{5.7}$$

is again an Azumaya algebra, with center $Z(A_1) \otimes_{Z(A)} Z(A_2)$.

Proof. Let C_i be the center of A_i, then as $A \otimes_C C_i$ is a C_i-Azumaya subalgebra of A_i it follows from the centralizer theorem that

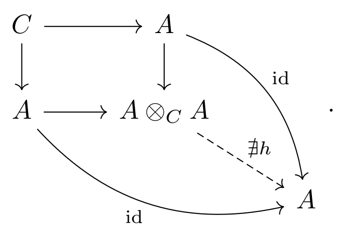
$$A_i \cong (A \otimes_C C_i) \otimes_{C_i} A_i^A \cong A \otimes_C A_i^A \tag{5.8}$$

But then we have the following isomorphisms.

$$\begin{aligned} A_1 \otimes_A A_2 &\cong A_1^A \otimes_C A \otimes_A A \otimes_C A_2^A \\ &\cong A_1^A \otimes_C A \otimes_C A_2^A \\ &\cong A_1 \otimes_C A_2^A \cong A_1^A \otimes_C A_2 \end{aligned}$$

As all A_i and A_i^A are separable over C (by transitivity of separability) it follows that $A_1 \otimes_C A_2^A$ and $A_1^A \otimes_C A_2$ are separable over C and hence are Azumaya algebras over their center. □

If a category $\mathcal{C}^{\mathrm{op}}$ has pullbacks (or, equivalently, the category $\mathcal{C}$ has pushouts) then one can restrict to a basis to define a Grothendieck topology on $\mathcal{C}^{\mathrm{op}}$. As we want to describe Grothendieck topologies on the (geometric) opposite category $\mathsf{Azu}^{\mathrm{op}}$, the previous result would be useful if the tensor product would be a pushout in Azu. However, this is *not* the case. Indeed, let A be an Azumaya algebra with center C and degree $n > 1$, then $A \otimes_C A$ is Azumaya of degree n^2 so cannot satisfy the condition for the diagram

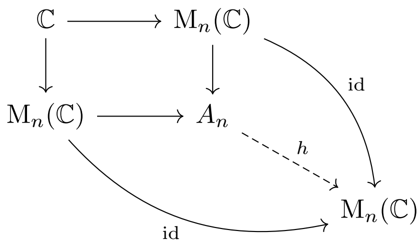


In fact, some diagrams in Azu cannot have *any* pushout.

Example 5.3. *Consider the diagram*

$$\begin{CD} \mathbb{C} @>>> \mathrm{M}_n(\mathbb{C}) \\ @VVV \\ \mathrm{M}_n(\mathbb{C}) \end{CD} .$$

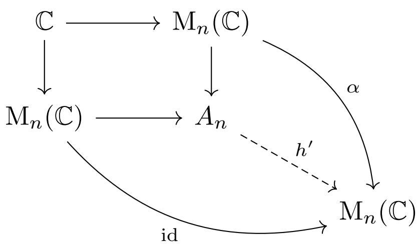
If the pushout of above diagram exists, then it is unique. Call it A_n *and write* $C = Z(A_n)$. *We will derive a contradiction from the existence of* A_n, *for* $n > 1$. *Consider the commutative diagram*



By definition of the pushout, the dashed arrow h *exists. It induces a morphism* $C \to \mathbb{C}$ *on centers, namely the unique morphism* $C \to \mathbb{C}$ *for which*

$$A_n \otimes_C \mathbb{C} \cong \mathrm{M}_n(\mathbb{C}).$$

This implies however that the dashed arrow h' *in*



still induces the same morphism on centers $C \to \mathbb{C}$, *for some nontrivial automorphism* α. *So* $h' = \beta \circ h$ *for some automorphism* β *of* $\mathrm{M}_n(\mathbb{C})$. *Commutativity of the diagram shows both* $\beta = \alpha$ *and* $\beta = \mathrm{id}$, *a contradiction.*

In this thesis, we always used Grothendieck topologies instead of Grothendieck pretopologies[1]. There are two main reasons:

- Grothendieck pretopologies can only be defined for categories with pullbacks;
- Grothendieck topologies on $\mathcal{C}$ are in bijection with subtoposes of $\mathsf{PSh}(\mathcal{C})$, see Proposition 1.6. There is no analogous result for Grothendieck pretopologies.

We saw above that the category Azu does not have pushouts, so $\mathsf{Azu}^{\mathrm{op}}$ does not have pullbacks. So in order to study subtoposes of $\mathsf{PSh}(\mathsf{Azu}^{\mathrm{op}})$, we need

[1] A Grothendieck pretopology is sometimes called a basis for a Grothendieck topology, see for example the definition in Mac Lane–Moerdijk [MLM94, Definition 2, p. 111].

to look at Grothendieck topologies in terms of sieves (instead of Grothendieck pretopologies). We will have the additional advantage that there is a bijection

$$\text{subtoposes of } \mathsf{PSh}(\mathsf{Azu}^{\mathrm{op}}) \leftrightarrow \text{Grothendieck topologies on } \mathsf{Azu}^{\mathrm{op}}. \tag{5.9}$$

Because the notations in $\mathsf{Azu}^{\mathrm{op}}$ are awkward, we will keep writing the arrows $A \to B$ between Azumaya algebras in the original, ring-theoretic direction. However, we have to keep in mind that the sieves, Grothendieck topologies and sheaves are all defined with respect to $\mathsf{Azu}^{\mathrm{op}}$, not with respect to Azu.

So a sieve on an Azumaya algebra A will be written as

$$L = \{f_i : A \to A_i\}_{i \in I}. \tag{5.10}$$

For $h : A \to B$ we get the inverse image sieve

$$h^{-1}L = \{g \ : \ g \circ h \in S\} \tag{5.11}$$

(where $\circ$ denotes composition in Azu).

We will relate Grothendieck topologies on $\mathsf{Azu}^{\mathrm{op}}$ to Grothendieck topologies on the full subcategory $\mathsf{Mat}^{\mathrm{op}}$ of $\mathsf{Azu}^{\mathrm{op}}$, and in this way to Grothendieck topologies on the big cell $\mathtt{D}$ as in Section 3.3. Here $\mathsf{Mat}^{\mathrm{op}}$ is the full subcategory given by the matrix algebras $\mathrm{M}_n(\mathbb{C})$ for $n \in \mathbb{N}_+$. The big cell $\mathtt{D}$ was introduced earlier: this is the category with a unique arrow

$$m \longrightarrow n$$

whenever $n \mid m$. Clearly, there is a projection $\pi : \mathsf{Mat} \to \mathtt{D}^{\mathrm{op}}$ sending a morphism $\mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_{nk}(\mathbb{C})$ to $n \to nk$.

Lemma 5.4. *Sieves on* $\mathrm{M}_n(\mathbb{C})$ *in* $\mathsf{Mat}^{\mathrm{op}}$ *are in bijection with sieves on* n *in* $\mathtt{D}$ *via* $L \mapsto \pi(L)$. *As a consequence, Grothendieck topologies on* $\mathsf{Mat}^{\mathrm{op}}$ *are in bijection with Grothendieck topologies on* $\mathtt{D}$.

Proof. The result follows if we can show that a sieve S on $\mathrm{M}_n(\mathbb{C})$ is fully determined by the multiples of n such that there is a morphism $\alpha : \mathrm{M}_n(\mathbb{C}) \longrightarrow \mathrm{M}_{nk}(\mathbb{C}) \in S$ and not on the actual morphism α. So, let $\beta : \mathrm{M}_n(\mathbb{C}) \longrightarrow \mathrm{M}_{nk}(\mathbb{C})$ be another morphism, then it follows from the double centralizer theorem that there is an automorphism γ of $\mathrm{M}_{nk}(\mathbb{C})$ such that $\gamma \circ \alpha = \beta$. But then we have

$$\alpha \in S \Leftrightarrow \beta \in S \tag{5.12}$$

from which the claims follow. □

For a description of Grothendieck topologies on the big cell, we refer to Section 3.3. In this Chapter, the Grothendieck topologies on $\mathtt{D}$ that are *of finite type* will play an important role. These are the Grothendieck topologies K_S as in (2.9) with $S \subseteq \mathbb{S}$ a patch, see Proposition 2.11.

Recall that $\{n_i \to n\}_{i \in I}$ is a K_S-covering sieve if for each $s \in S$ there is an $i \in I$ such that $n_i \mid s$. We recall some examples of patches. For the proofs, see Example 3.10.

- For $S = \mathbb{S}$ we get the minimal topology: only the maximal sieves are covering sieves. This is sometimes called the presheaf topology or the chaotic topology.

- For $S = \{\prod_p p^\infty\}$ we get the maximal topology: every non-empty sieve is a covering sieve. This is sometimes called the atomic topology.
- More generally, for $S = \{s\}$ there are two cases. If $n \nmid s$ then any sieve on n is a covering sieve, including the empty sieve. If $n \mid s$ then $\{n_i \to n\}_{i \in I}$ is a covering sieve if there is an $i \in I$ with $n_i \mid s$. These topologies will play a role in the last section of this chapter.

For more examples, see Example 3.10.

Definition 5.5. *Let $K = K_S$ be a Grothendieck topology of finite type on* $\mathtt{D}$*. Then we say that K is* stable under multiplication *if for any $k \in \mathbb{N}_+$*

$$\{n_i \to n\}_{i \in I} \text{ is a covering sieve } \Rightarrow \{kn_i \to kn\} \text{ is a covering sieve.} \quad (5.13)$$

We can reformulate this in terms of S.

Lemma 5.6. *Let $K = K_S$ be a Grothendieck topology of finite type on* $\mathtt{D}$*. Then K is stable under multiplication if and only if*

$$ks \in S \Rightarrow s \in S \quad (5.14)$$

for all $k \in \mathbb{N}_+$.

Proof. For $k \mid s$ with $k \in \mathbb{N}_+$, there is a unique supernatural number $\frac{s}{k}$ such that $k(\frac{s}{k}) = s$.

We show the "if" direction, the other direction is similar. Let $\{n_i \to n\}_{i \in I}$ be a covering sieve for K_S. Then we want to show that $\{kn_i \to kn\}_{i \in I}$ is a covering sieve as well, for $k \in \mathbb{N}_+$. It is enough to show that $kn \mid s \Rightarrow \exists i \in I,\ kn_i \mid s$, for all $s \in S$. So let $kn \mid s$. Then $n \mid \frac{s}{k}$ so there is an $i \in I$ such that $n_i \mid \frac{s}{k}$. But this means $kn_i \mid s$. □

Now we are ready to construct Grothendieck topologies on $\mathsf{Azu}^{\mathrm{op}}$.

Let J be a Grothendieck topology on $\mathsf{Comm}^{\mathrm{op}}$ and let K be a Grothendieck topology on $\mathtt{D}$. Let A be an Azumaya algebra with center C, and take a sieve

$$L = \{A \to A_i\}_{i \in I}. \quad (5.15)$$

Let $f : C \to D$ be a morphism of commutative rings.

Definition 5.7. *Take J, K, L and f be as above. Then we say that $m \in \mathbb{N}_+$ is* represented *on f if L contains a central extension of $A \otimes_C D$ that is of constant degree m over its center. If we can find such a central extension by a matrix algebra, then we say that m is* represented by a matrix algebra *on f. We say that f is* centrally covered *if $A \otimes_C D$ is of constant degree n, and the represented numbers on f form a K-covering sieve on n. The set of centrally covered morphisms will be denoted by*

$$\pi_K(L) = \{f : C \to D \text{ such that } f \text{ is centrally covered w.r.t. } K \text{ and } L\}. \quad (5.16)$$

Similarly, we say that f is centrally covered by matrix algebras *if $A \otimes_C D$ is isomorphic to a matrix algebra of constant degree n, and the numbers that are represented by a matrix algebra on f, form a K-covering sieve on n. The set of morphisms that are centrally covered by matrix algebras, will be denoted by*

$$\Pi_K(L) = \left\{ \begin{smallmatrix} f\, :\, C \to D \text{ such that } f \text{ is centrally} \\ \text{covered by matrix algebras w.r.t. } K \text{ and } L \end{smallmatrix} \right\}. \quad (5.17)$$

Definition 5.8. *Consider (J, K) with J a Grothendieck topology on $\mathsf{Comm}^{\mathrm{op}}$ and K a Grothendieck topology on $\mathtt{D}$. Let $L = \{A \to A_i\}_{i \in I}$ be a sieve on A. Then we say that L is a J_K-covering sieve if $\pi_K(L)$ as above is a J-covering sieve.*

We want to prove that under certain conditions the collection of J_K-covering sieves is a Grothendieck topology on $\mathsf{Azu}^{\mathrm{op}}$.

Theorem 5.9. *Let J be a Grothendieck topology on $\mathsf{Comm}^{\mathrm{op}}$ and K a Grothendieck topology on $\mathtt{D}$. Suppose that K is of finite type, so $K = K_S$ for a patch $S \subseteq \mathbb{S}$. Further, suppose that*
(a) J is finer than the étale topology, or
(b) J is finer than the Zariski topology and K is stable under multiplication.
Then the collection of J_K-covering sieves of Definition 5.8 defines a Grothendieck topology on $\mathsf{Azu}^{\mathrm{op}}$.

Proof. We prove the three axioms for a Grothendieck topology, see Section 1.1.

(GT1). Let A be an Azumaya algebra with center C. Then the maximal sieve L on A is a J_K-covering sieve. Indeed, let $f : C \to D$ be any morphism of commutative rings such that $A \otimes_C D$ is of constant degree. Then f is centrally covered, i.e. $f \in \pi_K(L)$. Now it is clear that $\pi_K(L)$ contains a Zariski covering, so in particular it is a J-covering sieve.

(GT2). Let $L = \{A \to A_i\}_{i \in I}$ be a J_K-covering sieve, and let $\varphi : A \to A'$ be a morphism of Azumaya algebras, inducing a morphism $\varphi_0 : C \to C'$ on centers. We need to show that $\varphi^{-1}L$ is again a J_K-covering sieve.

Case (a). We will first show that, in this case, $\Pi_K(L)$ is a J-covering sieve. It is enough to show that $f^{-1}\Pi_K(L)$ is a J-covering sieve, for each $f \in \pi_K(L)$. So take such a centrally covered morphism $f : C \to D$. The represented numbers on f form a K-covering sieve, and we can take a finitely generated covering sieve $(m_1, \ldots, m_k)$ contained in it, with each m_i represented by an Azumaya algebras A_i with center D. Take an étale cover $g : D \to E$ trivializing $A_1, \ldots, A_k$. Then g is contained in $f^{-1}\Pi_K(L)$, so $f^{-1}\Pi_K(L)$ is an étale covering sieve, so in particular it is a J-covering sieve.

Now take a morphism $g : C' \to D$ such that $g \circ \varphi_0$ is centrally covered by matrix algebras, and such that $A' \otimes_{C'} D$ is isomorphic to a matrix algebra. Then g is itself centrally covered with respect to $\varphi^{-1}L$. So there is an inclusion

$$M \cap \varphi_0^{-1}\Pi_K(L) \subseteq \Pi_K(\varphi^{-1}L) \tag{5.18}$$

where M is an étale covering sieve trivializing A'. This shows that $\Pi_K(\varphi^{-1}L)$ is a J-covering sieve, so $\varphi^{-1}L$ is a J_K-covering sieve.

Case (b). For this case, let M be a Zariski covering sieve on C' such that for $g : C' \to D$ in M we have that $A \otimes_C D$ and $A' \otimes_{C'} D$ are of constant degree. Then because K is stable under multiplication, it is easy to see that

$$M \cap \varphi_0^{-1}\pi_K(L) \subseteq \pi_K(\varphi^{-1}L). \tag{5.19}$$

(use the tensor product $- \otimes_A A'$). So $\varphi^{-1}L$ is a J_K-covering sieve.

(GT3). Let M be a J_K-covering sieve and suppose that $h^{-1}L$ is a J_K-covering sieve for all $h \in M$. We need to show that L is a J_K-covering sieve.

Take $f : C \to D$ with $f \in \pi_K(M)$. The representable numbers on f form a K-covering sieve, so take a finitely generated covering sieve $(m_1, \ldots, m_k)$

contained in it, with each m_i represented by an Azumaya algebra A_i with center D. For the corresponding morphisms $f_i : A \to A_i$ we have that $\pi_K(f_i^{-1}L)$ is a J-covering sieve. Moreover, if for each $i \in \{1, \dots, k\}$, $g : D \to E$ is centrally covered w.r.t. $f_i^{-1}L$, then $g \circ f$ is centrally covered w.r.t. L. In other words, we have an inclusion

$$\bigcap_{i=1}^{m} \pi_K(f_i^{-1}L) \subseteq f^{-1}\pi_K(L) \tag{5.20}$$

and because $\bigcap_{i=1}^{m} \pi_K(f_i^{-1}L)$ is a J-covering sieve, this shows that $f^{-1}\pi_K(L)$ is a J-covering sieve too. Moreover, this holds for every $f \in \pi_K(M)$. We conclude that $\pi_K(L)$ is a J-covering sieve, i.e. L is a J_K-covering sieve. □

Proposition 5.10. *Assume that J_K is not discrete. Then we can recover J as the collection of sieves $Z(L) = \{C \to Z(A_i)\}_{i \in I}$ for each sieve $L = \{C \to A_i\}_{i \in I}$ in J_K.*

Similarly, we can recover K as the collection of sieves $\deg(L) = \{n \to n_i\}_{i \in I}$ *for each sieve $S = \{\mathrm{M}_n(\mathbb{C}) \to A_i\}_{i \in I}$ in J_K with A_i of constant degree n_i.*

Proof. Let $L = \{C \to A_i\}$ be a J_K-covering sieve. Because K can not be the discrete topology, a morphism $f : C \to D$ can only be centrally covered if S contains at least one central extension of D. So we have

$$\pi_K(S) \subseteq Z(L) \tag{5.21}$$

and this shows that $Z(L)$ is a J-covering sieve. Conversely, let M be a J-covering sieve on C and consider the sieve $\langle M \rangle$ on $\mathsf{Azu}^{\mathrm{op}}$ that is generated by it. Because J can not be the discrete topology, M and $\langle M \rangle$ are both non-empty. So $\pi_K(\langle M \rangle) = Z(\langle M \rangle) = M$, which shows that M comes from some J_K-covering sieve by taking centers.

Let $L = \{\mathrm{M}_n(\mathbb{C}) \to A_i\}$ be a J_K-covering sieve. There is at least one morphism $\mathbb{C} \to D$ that is centrally covered, because J can not be the discrete topology. This shows that $\deg(L)$ is a K-covering sieve. Conversely, if $M = \{n \to n_i\}$ is a K-covering sieve, then it is by the assumption non-empty, and the sieve $\langle M' \rangle$ generated by $M' = \{\mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_{n_i}(\mathbb{C})\}$ is clearly a J_K-covering sieve with $\deg(\langle M' \rangle) = M$. □

Definition 5.11. *Consider (J, K) with J a Grothendieck topology on $\mathsf{Comm}^{\mathrm{op}}$ and K a Grothendieck topology of finite type on D. If*

(a) J is finer than the étale topology, or

(b) J is finer than the Zariski topology and K is stable under multiplication,

then J_K will be called a combined Grothendieck topology *on $\mathsf{Azu}^{\mathrm{op}}$. It is a Grothendieck topology by Theorem 5.9. Moreover, $J_K = J'_{K'}$ implies that either $J_K = J'_{K'}$ is discrete or we have $J = J'$ and $K = K'$.*

Example 5.12. *The minimal topology is stable under multiplication, so for different choices of J we get for example the minimal Zariski topology, the minimal étale topology or the minimal flat topology.*

Similarly, we can define the maximal Zariski topology, the maximal étale topology or the maximal flat topology.

The *maximal flat topology* will be important in the next section.

5.2 Azumaya representation schemes

Consider a finitely presented $\mathbb{C}$-algebra R and the corresponding set-valued functor

$$\begin{array}{ccc} \mathsf{Azu} & \longrightarrow & \mathsf{Sets} \\ A & \longmapsto & \mathsf{Alg}(R, A) \end{array}, \tag{5.22}$$

which we will denote by $\mathsf{Alg}(R, -)$. In this section, we will show that this functor is in fact a J_K-sheaf with J the flat topology and K the maximal topology (we say that the functor is a sheaf for the maximal flat topology).

Recall that the flat topology on $\mathsf{Comm}^{\mathrm{op}}$ is defined as follows. A morphism $\varphi : C \to D$ of finitely generated commutative rings is called *faithfully flat* if it has the property that a short exact sequence of C-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \tag{5.23}$$

is exact if and only if the corresponding short exact sequence of D-modules

$$0 \longrightarrow M' \otimes_C D \longrightarrow M \otimes_C D \longrightarrow M'' \otimes_C D \longrightarrow 0 \tag{5.24}$$

is exact. Now, by definition, a sieve is a covering sieve for the *flat topology* if it contains a family $\{C \to C_i\}_{i \in I}$ with

$$C \to \prod_{i \in I} C_i$$

faithfully flat.

When we show that $\mathsf{Alg}(R, -)$ is a sheaf for the maximal flat topology, this will imply that it is also a sheaf for any coarser Grothendieck topology, e.g. the Grothendieck topologies J_K as in the previous section where J is the Zariski or étale topology, and $K = K_S$ for a patch S containing $\prod_p p^\infty$. It immediately follows that for each Azumaya algebra A the set-valued functor

$$\begin{array}{ccc} \mathsf{Comm}_C & \longrightarrow & \mathsf{Sets} \\ D & \longmapsto & \mathsf{Alg}(R, A \otimes_C D) \end{array} \tag{5.25}$$

on the category of commutative C-algebras is also a sheaf for the flat topology.

This sheaf turns out to be representable by a scheme, which we will call the Azumaya representation scheme of R associated to A. We will give a ring-theoretic description of the coordinate ring of this scheme and discuss its geometric structure.

Lemma 5.13. *Let $A \to B$ be a morphism in* Azu*. Then the following are equivalent:*
(a) $A \to B$ is left faithfully flat;
(b) $A \to B$ is right faithfully flat;
(c) $Z(A) \to Z(B)$ is faithfully flat;
Moreover, if any of the above is satisfied, then the sequence

$$0 \longrightarrow A \longrightarrow B \rightrightarrows B \otimes_A B \tag{5.26}$$

is exact.

Proof. (1) ⇔ (3): by the Double Centralizer Theorem, the functor $-\otimes_A B$ is equivalent to $-\otimes_{Z(A)} Z(B)\otimes_{Z(B)} B^A$. Because B^A is always faithfully flat over its center $Z(B)$, we get that $A\to B$ is left faithfully flat if and only if $Z(A)\to Z(B)$ is faithfully flat.

(2) ⇔ (3): analogously.

The sequence in the lemma appeared in Artin [Art69] and is a noncommutative version of the Amitsur complex. By faithfully flatness, it is enough to check that

$$0 \longrightarrow B \xrightarrow{b\mapsto b\otimes 1} B\otimes_A B \underset{b\otimes b'\mapsto b\otimes 1\otimes b'}{\overset{b\otimes b'\mapsto b\otimes b'\otimes 1}{\rightrightarrows}} B\otimes_A B\otimes_A B \tag{5.27}$$

is exact. The morphism $B\to B\otimes_A B$ has a retraction given by the multiplication morphism. In particular it is injective. Further, suppose $\sum_i b_i\otimes b_i'\otimes 1 = \sum_i b_i\otimes 1\otimes b_i'$. Applying multiplication to the first two tensor factors, we get that $\sum b_i b_i'\otimes 1 = \sum_i b_i\otimes b_i'$. But this means that $\sum_i b_i\otimes b_i'$ lies in the image of $B\to B\otimes_A B$. □

Proposition 5.14. *The functor* $\mathsf{Alg}(R,-)$ *on* Azu *is a sheaf for the maximal flat topology on* $\mathsf{Azu}^{\mathrm{op}}$ *(and hence for any coarser Grothendieck topology).*

Proof. We need to prove that we can glue sections in a unique way whenever they agree locally. It is enough to show that

$$0 \longrightarrow \mathsf{Alg}(R,A) \longrightarrow \textstyle\prod_{i\in I}\mathsf{Alg}(R,A_i) \rightrightarrows \prod_{i,j\in I}\mathsf{Alg}(R,A_i\otimes_A A_j)$$

is exact for every family of morphisms $\{A\to A_i\}_{i\in I}$ in Azu such that $A\to\prod_{i\in I}A_i$ is faithfully flat (note that $\prod_{i\in I}A_i$ is not necessarily Azumaya). We know that $\mathsf{Alg}(R,-)$ commutes with limits of rings (in particular with categorical kernels, products and inverse limits), so it is enough to show that

$$0 \longrightarrow A \longrightarrow \textstyle\prod_{i\in J}A_i \rightrightarrows \prod_{i,j\in J}A_i\otimes_A A_j \tag{5.28}$$

is exact, for every finite subset $J\subseteq I$ such that $A\to\prod_{i\in J}A_i$ is still faithfully flat. But this follows from Lemma 5.13. □

If we fix a finitely generated Azumaya algebra A, we can consider the coslice category

$$\mathsf{Azu}_A \;=\; A\setminus\mathsf{Azu} \tag{5.29}$$

(see Subsection 1.3.4). The objects are finitely generated Azumaya algebras B with a center-preserving structure morphism $\varphi_B : A\to B$. The morphisms are center-preserving algebra morphisms $f: B\to B'$ such that $f\circ\varphi_B=\varphi_{B'}$.

We now have $\mathsf{Azu}_A^{\mathrm{op}} = \mathsf{Azu}^{\mathrm{op}}/A$. From Subsection 1.3.4 we know that any Grothendieck topology on $\mathsf{Azu}^{\mathrm{op}}$ lifts to a Grothendieck topology on $\mathsf{Azu}_A^{\mathrm{op}}$. The lift of a combined Grothendieck topology J_K on $\mathsf{Azu}^{\mathrm{op}}$ will still be called J_K.

Similarly, we define the category of finitely generated commutative C-algebras as $\mathsf{Comm}_C = C\setminus\mathsf{Comm}$. Grothendieck topologies on Comm will lift to Grothendieck topologies with the same name on Comm_C.

We consider the composition of geometric morphisms

$$\mathsf{Sh}(\mathsf{Comm}_C^{\mathrm{op}}) \longrightarrow \mathsf{Sh}(\mathsf{Azu}_C^{\mathrm{op}}) \longrightarrow \mathsf{Sh}(\mathsf{Azu}_A^{\mathrm{op}}) \longrightarrow \mathsf{Sh}(\mathsf{Azu}^{\mathrm{op}})\ , \tag{5.30}$$

with inverse image functors described as follows: for the middle arrow it is given by $F \mapsto F(A \otimes_C -)$ and the other inverse image functors are given by restriction. In each case, Sh is the category of sheaves for the (maximal) flat topology. The composition of the inverse image functors takes $\mathsf{Alg}(R, -)$ to the functor

$$\begin{aligned} \mathsf{Comm}_C &\longrightarrow \mathsf{Sets} \\ D &\longmapsto \mathsf{Alg}(R, A \otimes_C D) \end{aligned}$$

which is therefore also a sheaf for the flat topology. In the rest of the section, we will show that this sheaf is even representable by an affine scheme and describe its coordinate ring and basic properties.

For a $\mathbb{C}$-algebra S, *Artin S-bimodules* (see Artin [Art69] or Procesi [Pro73]) are vector spaces M equipped with compatible left and right S-action, and generated by invariants M^S as a two-sided S-module. *Artin S-algebras* are algebras R equipped with a structure morphism $\varphi_R : S \to R$ making R into an Artin bimodule. Equivalently, φ_R is a Procesi extension, see Procesi [Pro73]. We will denote by Bimod_S the category of Artin S-bimodules with morphisms that are S-linear on both sides. Similarly, Alg_S will denote the category of Artin S-algebras with S-linear algebra morphisms.

Now let C be a commutative algebra and A an Azumaya algebra over C. Note that this makes A into an Artin C-algebra. In Artin [Art69] it is shown that there are equivalences of categories

$$\mathsf{Bimod}_C \underset{(-)^A}{\overset{A \otimes_C -}{\rightleftarrows}} \mathsf{Bimod}_A \ , \tag{5.31}$$

$$\mathsf{Alg}_C \underset{(-)^A}{\overset{A \otimes_C -}{\rightleftarrows}} \mathsf{Alg}_A \ . \tag{5.32}$$

Observe that in the case of an Azumaya algebra A we can reformulate Artin's definition, by invoking the Double Centralizer Theorem. For an Azumaya A with center C, Artin A-bimodules are the ones such that the induced C-action is symmetric. Similarly, Artin A-algebras are the algebras with structure morphism sending C into the center.

In order to describe the functor $\mathsf{Alg}(R, A \otimes_C -)$, we have to introduce a generalization of the root algebra $\sqrt[n]{R}$, used in studying n-dimensional representations of R, see Bergman [Ber74] or Schofield [Sch85]. Note that morphisms $R \to A$ with A Azumaya over C are the same as C-algebra morphisms $R \otimes C \to A$, so we may assume that R is an Artin C-algebra.

Definition 5.15. *Let A be an Azumaya algebra with center C and let R be a C-algebra. Then the* A-th root algebra *of R, denoted $\sqrt[A]{R}$, is defined to be*

$$\sqrt[A]{R} = (R *_C A)^A. \tag{5.33}$$

Here $*_C$ *denotes the coproduct of* C*-algebras, i.e. the pushout of the diagram*

$$\begin{array}{ccc} C & \longrightarrow & R \\ \downarrow & & \\ A & & \end{array}$$

in the category of rings.

Proposition 5.16. *The functor* $\sqrt[A]{-} : \mathsf{Alg}_C \to \mathsf{Alg}_C$ *is left adjoint to tensoring* $- \otimes_C A : \mathsf{Alg}_C \to \mathsf{Alg}_C$.

Proof. Note that we can write the functor $A \otimes_C - : \mathsf{Alg}_C \to \mathsf{Alg}_C$ as a composition

$$\mathsf{Alg}_C \xrightarrow{A\otimes_C -} \mathsf{Alg}_A \longrightarrow \mathsf{Alg}_C, \tag{5.34}$$

where the first functor is the equivalence (5.32) and the second functor is the forgetful one. Being an equivalence, the first one has its quasi-inverse $(-)^A$ as left adjoint. Further, one can check that the second one has left adjoint $A *_C -$. The proposition follows from composition of adjunctions. □

Theorem 5.17. *If* A *an Azumaya algebra with center* C*, then for every Artin* C*-algebra* R *there is an affine* C*-scheme* $\mathrm{rep}_A(R)$*, which we call the* Azumaya representation scheme *of* R *with respect to* A*, representing the functor*

$$\mathsf{Comm}_C \longrightarrow \mathsf{Sets} \qquad D \mapsto \mathsf{Alg}_C(R, A \otimes_C D). \tag{5.35}$$

Proof. Define the Azumaya representation scheme as

$$\mathrm{rep}_A(R) = \mathrm{Spec}(\sqrt[A]{R})_{\mathrm{ab}}. \tag{5.36}$$

To check that this represents the given functor, use Proposition 5.16 and the fact that $- \otimes C$ and ab are both adjoint to the appropriate forgetful functors. □

Remark 5.18. *For a* $\mathbb{C}$*-algebra* S*, we will use the shorthand notation*

$$\mathrm{rep}_A(S) = \mathrm{rep}_A(S \otimes C). \tag{5.37}$$

Proposition 5.19. *Let* A *and* B *be Azumaya algebras with center* C*. Let* R *be a* C*-algebra and* S *a* $\mathbb{C}$*-algebra.*

(a) *There are natural isomorphisms* $\sqrt[A]{\sqrt[B]{R}} \simeq \sqrt[A\otimes_C B]{R} \simeq \sqrt[B]{\sqrt[A]{R}}$ *of* C*-algebras.*
(b) *For any morphism of commutative algebras* $C \to D$*, there are natural isomorphisms* $\sqrt[A\otimes_C D]{R \otimes_C D} \simeq \sqrt[A]{R} \otimes_C D$.
(c) *Suppose that* A *is of constant degree* n*. Then* $\sqrt[A]{S \otimes C}$ *is, étale locally on* C*, isomorphic to* $\sqrt[n]{S} \otimes C$.
(d) *A* C*-linear morphism* $A \to B$ *induces a* C*-linear morphism* $\sqrt[B]{R} \to \sqrt[A]{R}$*, functorial in* R.

Proof. All statements follow by invoking the Yoneda Lemma and some computations. We prove (1) as an example. For any C-algebra S, we have

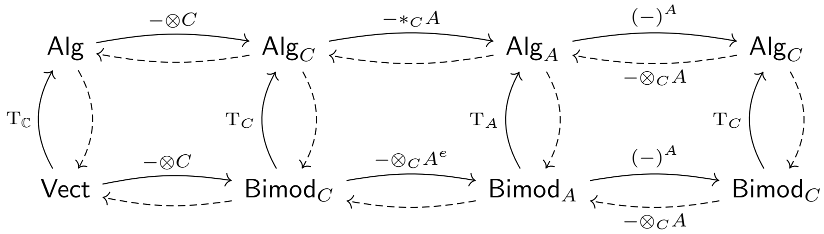
$$\begin{aligned} \mathsf{Alg}_C(\sqrt[A]{\sqrt[B]{R}}, S) &\simeq \mathsf{Alg}_C(\sqrt[B]{R}, A \otimes_C S) \\ &\simeq \mathsf{Alg}_C(R, B \otimes_C A \otimes_C S) \\ &\simeq \mathsf{Alg}_C(\sqrt[A\otimes_C B]{R}, S), \end{aligned}$$

so by the Yoneda Lemma we have $\sqrt[A]{\sqrt[B]{R}} \simeq \sqrt[A\otimes_C B]{R}$. Similarly for $\sqrt[B]{\sqrt[A]{R}}$. □

Note that, by part (c), $\mathrm{rep}_A(S)$ is étale locally on C isomorphic to $\mathrm{rep}_n(S) \times \mathrm{Spec}(C)$, for S a $\mathbb{C}$-algebra. So Azumaya representation schemes are *twisted* versions of representation schemes, similar to Azumaya algebras being twisted versions of matrix algebras.

Example 5.20. *Let A be an Azumaya algebra with center C. Then the C-linear automorphisms of A form a sheaf on* $\mathrm{Spec}(C)$*, which is represented by* $\mathrm{rep}_A(A)$*.*

Example 5.21 (Free algebras)**.** *Consider the diagram of adjunctions*



,

where the dashed arrows are right adjoint to the solid ones. The unlabeled functors are forgetful functors. It is obvious that the diagram of dashed arrows is commutative and by uniqueness of adjoint functors this implies that the diagram of solid arrows is commutative too. In particular we have

$$\sqrt[A]{(\mathrm{T}_{\mathbb{C}}V) \otimes C} \simeq \mathrm{T}_C\left((V \otimes A^e)^A\right) \simeq \mathrm{T}_C(V \otimes A^\vee) \tag{5.38}$$

for any vector space V. Here $A^\vee$ is the C-linear dual of A. More generally, for any C-module M we have

$$\sqrt[A]{\mathrm{T}_C M} \simeq \mathrm{T}_C(M \otimes_C A^\vee) \tag{5.39}$$

5.3 Trivializing Grothendieck topologies on Azumaya algebras

We will continue writing Comm for the category of finitely generated commutative rings, and Azu for the category of finitely generated Azumaya algebras and center-preserving algebra morphisms. So if A is a finitely generated Azumaya algebra with center C, and B is a finitely generated Azumaya algebra with center D, then a morphism in Azu is an algebra morphism

$$f : A \to B$$

such that $f(C) \subseteq D$. Each finitely generated Azumaya algebra has a finitely generated center. To see this, note that the center is $\mathbb{C}[\mathrm{rep}_n A]^{\mathrm{GL}_n}$ and use Hilbert's theorem on invariant subrings. Alternatively, use the (noncommutative) Artin–Tate lemma, see McConnell–Robson [MR01, Lemma 13.9.10]. So taking centers gives a functor

$$\mathrm{Z} : \mathsf{Azu} \longrightarrow \mathsf{Comm}. \tag{5.40}$$

Let J be a Grothendieck topology on $\mathsf{Comm}^{\mathrm{op}}$, and let K be a Grothendieck topology on $\mathtt{D}$.

We return to the combined Grothendieck topologies J_K as in Definition 5.11.

For some of the Grothendieck topologies J_K one could say that all Azumaya algebras are J_K-locally given by matrix algebras. We can make this idea precise in a few different ways, that turn out to be equivalent.

Proposition 5.22. *Let J_K be a combined Grothendieck topology. Then the following are equivalent:*

(a) for every Azumaya algebra A, there is a family of morphisms

$$\{A \to \mathrm{M}_{n_i}(C_i)\}_{i\in I} \tag{5.41}$$

generating a J_K-covering sieve;

(b) for every Azumaya algebra A, there is a family of morphisms

$$\{A \to \mathrm{M}_{n_i}(C_i)\}_{i\in I} \tag{5.42}$$

such that the induced morphism

$$\coprod_{i\in I} \mathsf{Azu}(\mathrm{M}_{n_i}(C_i), -) \longrightarrow \mathsf{Azu}(A, -) \tag{5.43}$$

is a J_K-epimorphism (after sheafification);

(c) for every Azumaya algebra A and J_K-covering sieve L on A, the sieve L contains a J_K-covering sieve generated by a family

$$\{A \to \mathrm{M}_{n_i}(C_i)\}_{i\in I}. \tag{5.44}$$

Proof. (3) ⇒ (1) ⇔ (2) This is clear.

(1) ⇒ (3) Suppose $L = \{A \to A_i\}_{i\in I}$. Then take for each A_i a family

$$\{A_i \to \mathrm{M}_{n_{ir}}(C_{ir})\}_{r\in R_i}$$

generating a J_K-covering sieve. Then $\{A \to \mathrm{M}_{n_{ir}}(C_{ir})\}_{r\in R_i}$ generates a J_K-covering sieve on A. □

Definition 5.23. *A combined Grothendieck topology J_K satisfying the equivalent conditions above will be called* trivializing.

Note that J_K is always trivializing whenever J is finer than the étale topology. If J is the Zariski topology, then we need more restrictions on the Grothendieck topology K.

Example 5.24. *Let $A = (x, y)_n$ be the cyclic algebra on $C = \mathbb{C}[x, y, x^{-1}, y^{-1}]$, then it is of degree n and of period n (the* period *is the order in the Brauer group). Moreover, because $\mathbb{C}[x, y, x^{-1}, y^{-1}]$ is the coordinate ring of a nonsingular and connected variety, the period of A is still n after Zariski localization: the restriction morphism $\mathrm{Br}(\mathbb{C}[x^{\pm}, y^{\pm}]) \to \mathrm{Br}(\mathbb{C}(x, y))$ is injective. So if $f : C \to D$ is some Zariski localization, and $A \otimes_C D \to \mathrm{M}_k(D)$ is a center-preserving morphism, then $\mathrm{M}_k(D) \cong (A \otimes_C D) \otimes_D B$ for some Azumaya algebra B. This B is of opposite Brauer class, so it has again period n. Because the period divides the degree, we find $n^2 \mid k$. So let $K = K_S$ for some patch $S \subseteq \mathbb{S}$. Then in order for Zar_K to be trivializing, we need $n \mid s \Rightarrow n^2 \mid s$ for all $s \in S$. In other words, S contains only supernatural numbers of the form*

$$s_\Sigma = \prod_{p\in\Sigma} p^\infty \tag{5.45}$$

with Σ some set of primes. In the special case $\Sigma = \varnothing$ we set $s_\varnothing = 1$. The supernatural numbers of the form s_Σ will be called completely infinite.

Conversely, suppose that $K = K_S$ with $S \subseteq \mathbb{S}$ a patch consisting of only completely infinite supernatural numbers as in the example. Then for all $n, k \in \mathbb{N}_+$ we have that $\{n^k \to n\}$ generates a K-covering sieve. We will use a result of Bass which says that every Azumaya algebra of degree n can be embedded into a matrix algebra of degree n^k for some k, see Proposition 6.1 and the remark afterwards in Bass–Roy [BR67], Chapter 1. Now take A an Azumaya algebra with center C and let $f : C \to D$ be a morphism such that $A \otimes_C D$ is of constant degree n. Then we can take a central extension $A \otimes_C D \subseteq \mathrm{M}_{n^k}(D)$ for some $k \in \mathbb{N}_+$. The sieve generated by these central extensions for each such f is clearly a J_K-covering sieve, so J_K is trivializing. We summarize the above in the following proposition.

Proposition 5.25. *Let* Zar *be the Zariski topology and consider $K = K_S$ with $S \subseteq \mathbb{S}$ a patch. Then Zar_K is trivializing if and only if S only contains* completely infinite *supernatural numbers, i.e. if and only if each $s \in S$ can be written as*

$$s = \prod_{p \in \Sigma} p^\infty \tag{5.46}$$

for some set Σ of primes.

For any patch $S \subseteq \mathbb{S}$, the subset of completely infinite elements is again a patch: it is the intersection of S with $2^{\mathcal{P}}$ from Example 3.10. In this way, we can easily construct trivializing Grothendieck topologies from Grothendieck topologies that are not trivializing.

5.4 Topos-theoretic points

From Subsection 1.3.2, we know that the points of $\mathsf{PSh}(\mathsf{Azu}^{\mathrm{op}})$ correspond precisely to the ind-objects in Azu. Moreover, this category of ind-objects can be embedded fully faithfully into the category ZAlg of all algebras and center-preserving algebra morphisms. Indeed, we have a natural isomorphism

$$\mathsf{ZAlg}(\varinjlim_i A_i, \varinjlim_j B_j) \simeq \varprojlim_i \varinjlim_j \mathsf{Azu}(A_i, B_j), \tag{5.47}$$

because each A_i and B_j is finitely generated.

Definition 5.26. *An* ind-Azumaya algebra *is an algebra that can be written as a filtered colimit of Azumaya algebras (with center-preserving transition maps). The* category of ind-Azumaya algebras *has as objects the ind-Azumaya algebras and as morphisms the center-preserving algebra morphisms.*

One notable area where inductive limits of matrix algebras play a role, is the theory of C^*-algebras. Recall that a *UHF-algebra* is the closure of

$$\bigcup_{i \in \mathbb{N}} \mathrm{M}_{n_i}(\mathbb{C}) \tag{5.48}$$

for some chain

$$\mathrm{M}_{n_1}(\mathbb{C}) \to \mathrm{M}_{n_2}(\mathbb{C}) \to \cdots \to \mathrm{M}_{n_i}(\mathbb{C}) \to \dots \tag{5.49}$$

where the morphisms are algebra morphisms (so they are injective). It is well-known that UHF-algebras are classified by their associated supernatural number s, which is uniquely determined by

$$n \mid s \;\Leftrightarrow\; \exists i \in \mathbb{N} \text{ with } n \mid n_i. \tag{5.50}$$

One can check that the same classification holds for the algebras of the form $\bigcup_{i\in\mathbb{N}} \mathrm{M}_{n_i}(\mathbb{C})$, i.e. without taking the closure, so we will denote these by $\mathrm{M}_s(\mathbb{C})$, where s is the associated supernatural number.

Remark 5.27. *Note that not every ind-Azumaya with center $\mathbb{C}$ is a union of a countable chain of matrix algebras. If $\{V_i\}_{i\in I}$ is an infinite set of finite dimensional vector spaces, then the matrix algebras*

$$\mathrm{End}\left(\bigotimes_{j\in J} V_j\right) \tag{5.51}$$

for $J \subset I$ finite, form a filtered diagram. It is easy to see that these often do not correspond to UHF-algebras.

An ind-Azumaya algebra B is a point for $\mathsf{Sh}(\mathsf{Azu}^{\mathrm{op}}, J_K)$ if and only if it is J_K-local, i.e. if and only if for each $A \to B$ and covering sieve $\{A \to A_i\}_{i\in I}$ there is an $i \in I$ and a factorization

$$\begin{array}{ccc} & A_i & \\ \nearrow & & \searrow \\ A & \longrightarrow & B \end{array} .$$

But before we can say anything about the points of the Azumaya toposes, we need the following proposition, which is straightforward but was not found in the literature.

Proposition 5.28. *Let J be a Grothendieck topology on $\mathsf{Comm}^{\mathrm{op}}$ with enough points. Take C in $\mathsf{Comm}^{\mathrm{op}}$ and let L be a sieve on C. Then L is a J-covering sieve if and only if for every J-local ring D and morphism $C \to D$ there is some $C \to C'$ in L and a factorization*

$$\begin{array}{ccc} & C' & \\ \nearrow & & \searrow \\ C & \longrightarrow & D \end{array} . \tag{5.52}$$

Proof. The "only if" part follows by definition of J-local ring, so we only need to prove the "if" part.

For $L = \{C \to C_i\}_{i\in I}$, consider the morphism

$$\coprod_{i\in I} \mathsf{Comm}(C_i, -) \xrightarrow{\xi} \mathsf{Comm}(C, -). \tag{5.53}$$

By definition, the stalk of $\mathsf{Comm}(C, -)$ at a J-local ring D is given by

$$\varinjlim_{j} \mathsf{Comm}(C, D_j) \tag{5.54}$$

where $D = \varinjlim_j D_j$ for $(D_j)_j$ is a filtered system of finitely generated commutative rings. But because C is itself finitely generated, we have an isomorphism $\varinjlim_j \mathsf{Comm}(C, D_j) \simeq \mathsf{Comm}(C, D)$. So the assumption implies that ξ is an epimorphism (this can be checked on stalks because J has enough points). In particular, the identity morphism $C \to C$ is locally in the image of ξ, and this means that the maps in the image of ξ form a covering sieve. But these are exactly the maps in L, so L is a J-covering sieve. □

Note that in the above proposition we explicitly use that the objects of the site are *finitely generated* commutative rings.

Theorem 5.29. *Let J_K be a trivializing combined Grothendieck topology, so $K = K_S$ for some patch $S \subseteq \mathbb{S}$. Then*

$$\mathrm{P}(J, K) = \{\mathrm{M}_s(D) \mid s \in S \text{ and } D \text{ a } J\text{-local commutative algebra}\} \tag{5.55}$$

is a family of points for $\mathsf{Sh}(\mathsf{Azu}^{\mathrm{op}}, J_K)$. If J has enough points, then this is a separating family of points for J_K, so in this case J_K has enough points too.

Proof. We first show that each $\mathrm{M}_s(D)$ as above is J_K-local. Let A be an Azumaya algebra and take a morphism $A \to \mathrm{M}_s(D)$. Let L be a J_K-covering sieve on A. We can assume that L is generated by matrix algebras. Then the sieve of centrally covered morphisms $\pi_K(L)$ is a J-covering sieve on the center C of A. In particular, there is a centrally covered morphism $C \to C'$ and a factorization

$$\begin{array}{ccccc} & & C' & & \\ & \nearrow & & \searrow & \\ C & & \longrightarrow & & D \end{array} \quad . \tag{5.56}$$

Moreover, we can find a central extension $A \otimes_C C' \to A'$ with A' of degree $m \mid s$. Because L is generated by matrix algebras, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{M}_{m'}(C'') & \longrightarrow & A' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A \otimes_C C' \end{array} \tag{5.57}$$

with $m' \mid m \mid s$. The desired factorization is now given by a choice of dashed arrow

$$\begin{array}{ccccc} & & \mathrm{M}_{m'}(C'') & & \\ & \nearrow & & \searrow & \\ A & & \longrightarrow & & \mathrm{M}_s(D) \end{array} \quad , \tag{5.58}$$

which on centers is given by the composition $C'' \to C' \to D$.

It remains to show that the given family of points $\mathrm{P}(J, K)$ is separating whenever J has enough points. Take a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ inducing isomorphisms on stalks for all points in $\mathrm{P}(J, K)$. Consider $x, y \in \mathcal{F}(A)$ with $\varphi(x) = \varphi(y)$, with A a finitely generated Azumaya algebra. We claim that $x = y$. Let L be the sieve of morphisms $f : A \to A'$ such that $x = y$ after restriction along f. Then it is enough to show that L is a J_K-covering sieve. In particular, we can assume that A is of constant degree n.

Let $\mathrm{M}_s(D)$ be a J_K-local ring in $\mathrm{P}(J,K)$, and take a morphism $g : A \to \mathrm{M}_s(D)$. Write $\mathrm{M}_s(D) = \varinjlim_j B_j^s$ with $(B_j^s)_j$ a filtered system of (finitely generated) Azumaya algebras. We can find a j such that $g(A) \subseteq B_j^s$ and such that $g : A \to B_j^s$ is in L. Because D is Zariski-local, we can assume that B_j^s is of constant degree $n_s \mid s$ over its center D_s. We can do this for every $s \in S \cap (n)$, and then take a finite subset $S' \subseteq S \cap (n)$ such that $\{n_s \to n\}_{s \in S'}$ generates a K-covering sieve on n. Let D' be the tensor product over C of the centers D_s with $s \in S'$. Then the morphism $C \to D'$ is centrally covered w.r.t. L, and there is a factorization

$$\begin{array}{ccc} & D' & \\ \nearrow & & \searrow \\ C & \longrightarrow & D \end{array} \quad . \tag{5.59}$$

If the morphism $C \to D$ was arbitrary, this would prove that $\pi_K(L)$ is a J-covering sieve, by Proposition 5.28. So we still need to prove that for every $h : C \to D$ with D J-local and for every $s \in S \cap (n)$, we can find a morphism $g : A \to \mathrm{M}_s(D)$ inducing h on centers. Take a family of morphisms

$$\{A \to \mathrm{M}_{n_i}(C_i)\}_{i \in I}$$

generating a J_K-covering sieve. Then for an arbitrary $h : C \to D$ we can find a centrally covered morphism $C \to C'$ and a factorization

$$\begin{array}{ccc} & C' & \\ \nearrow & & \searrow \\ C & \xrightarrow{\;h\;} & D \end{array} \quad . \tag{5.60}$$

From this, it is easy to construct a morphism $A \to \mathrm{M}_s(D)$ inducing the given morphism on centers.

In order to prove that φ is surjective, we take $y \in \mathcal{G}(A)$ and consider the sieve L of morphisms $f : A \to A'$ such that y is in the image of φ after restriction along f. In the same way as above, we can show that L is a J_K-covering sieve. Because φ is injective, the preimages are unique so they can be glued to a preimage of y. □

Note that the J-local commutative algebras are known in a lot of interesting cases, including the Zariski and étale topology. For an overview, we refer to Gabber–Kelly [GK15, Table 1]. In [GK15, Lemma 3.3], some properties of points for the flat (fppf) topology are given, but this case is more difficult and still lacks a concrete description. Schröer in [Sch17] also studies points for the fppf topology, but he considers the category of arbitrary commutative rings instead of only finitely generated ones.

Suppose that every finitely generated commutative algebra C is compact with respect to the given Grothendieck topology J, i.e. any J-covering sieve can be refined to a finitely generated J-covering sieve. Then $\mathsf{Sh}(\mathsf{Comm}^{\mathrm{op}}, J)$ is a coherent topos, so it has enough points by Deligne's completeness theorem. Examples are the Zariski, Nisnevich, étale and flat topology.

As in Theorem 5.29, let J_K be a trivializing combined Grothendieck topology, $K = K_S$ for some patch $S \subseteq \mathbb{S}$. If J_K is moreover coarser than the maximal flat topology, then $\mathsf{Alg}(R, -)$ is a J_K-sheaf for every algebra R, see Section 5.2. We would like to determine topos-theoretic points of the slice topos

$$\mathsf{Sh}(\mathsf{Azu}^{\mathrm{op}}, J_K)/\mathsf{Alg}(R, -) \tag{5.61}$$

As in Subsection 1.3.4 we can take a point p of $\mathsf{Sh}(\mathsf{Azu}^{\mathrm{op}}, J_K)$ and an element $x \in p^*\mathsf{Alg}(R, -)$, and construct the point (p, x) for the slice topos

$$\mathsf{Sh}(\mathsf{Azu}^{\mathrm{op}}, J_K)/\mathsf{Alg}(R, -). \tag{5.62}$$

Here

$$(p, x)^*\mathcal{G} = \varphi_p^{-1}(x) \tag{5.63}$$

where $\varphi_p : p^*\mathcal{G} \longrightarrow p^*\mathsf{Alg}(R, -)$ is the map induced by the structure morphism $\mathcal{G} \longrightarrow \mathsf{Alg}(R, -)$.

If we let p vary over the family $\mathrm{P}(J, K)$ of Theorem 5.29, we get the family of points

$$\mathrm{P}_R(J, K) = \{R \to \mathrm{M}_s(D) \mid s \in S \text{ and } D \text{ a } J\text{-local commutative algebra}\},$$

for the slice topos. Here we assume that J_K is a trivializing combined Grothendieck topology, coarser than the maximal flat topology, with S the patch associated to K. If J has enough points, then the set $\mathrm{P}_R(J, K)$ is separating.

5.5 Sheaves with action of the projective general linear group

Let $s \in \mathbb{S}$ be a supernatural number. Then the singleton $\{s\} \subseteq \mathbb{S}$ is a patch, and we can look at the corresponding Grothendieck topology K_s. We will use the shorthand J_s for the Grothendieck topology $J_{(K_s)}$ on $\mathsf{Azu}^{\mathrm{op}}$. We will always assume that J_s is a trivializing combined Grothendieck topology. By the results of Section 5.3, this is the case for example when

(a) J is finer than the étale topology and s is arbitrary, or

(b) J is finer than the Zariski topology and s is completely infinite, i.e. it is of the form

$$s = \prod_{p \in \Sigma} p^\infty \tag{5.64}$$

for Σ some set of primes.

These are the two cases we keep in mind for this section. Moreover, we will often assume that every J-covering sieve contains a finitely generated J-covering sieve.

Now consider A a finitely generated Azumaya algebra of constant degree n over its center C.

(a) If $n \nmid s$, then every sieve on A is a J_s-covering sieve, including the empty sieve.

(b) If $n \mid s$, then a sieve L on A is a covering sieve if and only if there is some J-covering $\{f_i : C \to C_i\}_{i \in I}$ such that for each $i \in I$, the sieve L contains a composition

$$\begin{array}{ccc} & & B_i \\ & & \uparrow{\scriptstyle \varphi_i} \\ A & \xrightarrow{1\otimes f_i} & A \otimes_C C_i \end{array} \tag{5.65}$$

with φ_i some central extension, B_i of degree $m \mid s$ over its center C_i.

One important case for J_s was already introduced in Section 5.1: if we take

$$s = \prod_p p^\infty \tag{5.66}$$

then J_s is the maximal topology $J_{\max}$ associated to a Grothendieck topology J on $\mathsf{Comm}^{\mathrm{op}}$.

In Section 5.2, we saw that the functor

$$\begin{gathered} \mathsf{Alg}(R,-) : \mathsf{Azu} \longrightarrow \mathsf{Sets} \\ A \mapsto \mathsf{Alg}(R,A) \end{gathered}$$

is a sheaf for the maximal flat topology on $\mathsf{Azu}^{\mathrm{op}}$, i.e. for $J_{\max}$ with J the flat topology (or any coarser topology). Similarly, the representable presheaves $\mathsf{Azu}(A,-)$ are sheaves for the maximal flat topology (because morphisms are center-preserving if and only if they are center-preserving $J_{\max}$-locally). In particular, $J_{\max}$ is a subcanonical Grothendieck topology for J the flat topology.

We will now show that we can easily construct J_s-sheaves from $J_{\max}$-sheaves, with s any supernatural number.

Let A be a finitely generated Azumaya algebra with center C. Then the degree of A is locally constant over $\mathrm{Spec}(C)$, so there is a unique decomposition

$$C = C_1 \times \cdots \times C_k \tag{5.67}$$

into components, such that $A \otimes_C C_i$ is of constant rank d_i for $i = 1, \ldots, k$ and such that $d_i \neq d_j$ for $i \neq j$. If s is a supernatural number, we then define the *s-truncation of A* to be the Azumaya algebra

$$A_s = \prod_{d_i \mid s} A \otimes_C C_i \tag{5.68}$$

(the empty product gives the zero ring). Similarly, we define the *s-truncation of a presheaf $\mathcal{F}$* to be

$$\mathcal{F}_{\downarrow s}(A) = \mathcal{F}(A_s). \tag{5.69}$$

One can check that this is again a presheaf.

Proposition 5.30. *Let $\mathcal{F}$ be a $J_{\max}$-sheaf, and let s be a supernatural number. Then $\mathcal{F}_{\downarrow s}$ is a J_s-sheaf, and it is naturally isomorphic to the J_s-sheafification of $\mathcal{F}$.*

Proof. Let A be an Azumaya algebra of constant rank n. If $n \nmid s$, then

$$\mathcal{F}_{\downarrow s}(B) = \{*\}$$

whenever there is a morphism $A \to B$. So, for trivial reasons, $\mathcal{F}_{\downarrow s}$ satisfies the gluing criterion with respect to J_s-covering sieves on A. If $n \mid s$, then any

J_s-covering sieve on A can be refined to a covering sieve generated by Azumaya algebras with degrees dividing s. Such a sieve is a $J_{\max}$-covering sieve as well, so here the gluing criterion is satisfied because $\mathcal{F}$ is a $J_{\max}$-sheaf. If A is not of constant rank, then we can take a Zariski cover on which it is. So this case reduces to the previous two cases.

This shows that $\mathcal{F}_{\downarrow s}$ is a J_s-sheaf.

Take an arbitrary J_s-sheaf $\mathcal{G}$. The morphism $\pi : A \to A_s$ generates a J_s-covering sieve, so the induced map $\mathcal{G} \to \mathcal{G}_{\downarrow s}$ is an isomorphism. Now it is easy to show that $\mathcal{F} \mapsto \mathcal{F}_{\downarrow s}$ is the sheafification functor. □

The next goal of this section is to describe the topos

$$\mathsf{Sh}(\mathsf{Azu}^{\mathrm{op}}, J_s) \tag{5.70}$$

in terms of sheaves on $\mathsf{Comm}^{\mathrm{op}}$ equipped with an action of an (infinite) projective linear group PGL_s, that we will define below. The inspiration for this equivalence comes from Caramello [Car16], where atomic toposes are described in terms of a topological group, under some mild assumptions. Because $\mathsf{Sh}(\mathsf{Azu}^{\mathrm{op}}, J_s)$ is not an atomic topos, the results from Caramello [Car16] do not easily transfer to our setting. However, the general principle still works, and this will result in Theorem 5.34.

For each $s \in \mathbb{S}$, we would like to define PGL_s as

$$\mathrm{PGL}_s(C) = \bigcup_{n|s} \mathrm{Aut}_C(\mathrm{M}_n(C)), \tag{5.71}$$

but we have to take a uniform choice as to how the automorphism groups are embedded into each other. So we define a functor

$$\mathrm{M} : \mathsf{Comm} \times \mathtt{D}^{\mathrm{op}} \longrightarrow \mathsf{Azu} \tag{5.72}$$

as follows. For each $n \in \mathbb{N}_+$, consider the matrix algebra

$$\mathrm{M}(C, n) = \mathrm{M}_{p_1}(C) \otimes_C \cdots \otimes_C \mathrm{M}_{p_k}(C) \tag{5.73}$$

with $p_1 \leq \cdots \leq p_k$ the prime factors of n, each occurring with the right multiplicity. Moreover, for each $n \mid m$ and morphism $C \to D$, we consider the morphism

$$\mathrm{M}(C, n) = \mathrm{M}_{p_1}(C) \otimes_C \cdots \otimes_C \mathrm{M}_{p_k}(C) \xrightarrow{\rho_{n,m}} \mathrm{M}_{q_1}(D) \otimes_D \cdots \otimes_D \mathrm{M}_{q_l}(D) = \mathrm{M}(D, m) \tag{5.74}$$

which is given by sending the k-th occurrence of a tensor factor $\mathrm{M}_p(C)$ on the left to the k-th occurrence of $\mathrm{M}_p(D)$ on the right side. It is easy to see that this turns M into a functor. In the following, we make the identification $\mathrm{M}_n(C) = \mathrm{M}(C, n)$, and the inclusion $\rho_{n,m}$ will be called a *standard embedding*.

Note that $\rho_{n,m}$ as above induces a morphism

$$\mathrm{Aut}_C(\mathrm{M}_n(C)) \longrightarrow \mathrm{Aut}_C(\mathrm{M}_m(D)). \tag{5.75}$$

We then define the *projective general linear group of degree s* as the union

$$\mathrm{PGL}_s(C) = \bigcup_{n|s} \mathrm{Aut}_C(\mathrm{M}_n(C)). \tag{5.76}$$

Remark 5.31. *In algebraic K-theory, there are at least two different ways to define infinite general linear groups, corresponding to different ways of embedding matrix algebras into each other. You can consider embeddings of the form* $\mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_{n+1}(\mathbb{C})$ *where each* $A \in \mathrm{M}_n(\mathbb{C})$ *is sent to the block matrix*

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{M}_{n+1}(\mathbb{C}). \tag{5.77}$$

The resulting infinite general linear group is denoted by $GL(\mathbb{C})$ *in Bass–Roy [BR67]. Note that the embeddings* $\mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_{n+1}(\mathbb{C})$ *are not ring morphisms (the zero element is not preserved). Another way is to embed matrix algebras using ring morphisms, like in this paper. The notation in [BR67] for the resulting infinite group is then* $GL_{\otimes}(\mathbb{C})$.

Note that there is an obvious action of $\mathrm{PGL}_s(C)$ on the infinite matrix algebra

$$\mathrm{M}_s(C) = \bigcup_{n|s} \mathrm{M}_n(C). \tag{5.78}$$

where the inclusions are again the standard embeddings $\rho_{n,m}$. Using this action, we can define *equivalence relations* $\sim_n$ on $\mathrm{PGL}_s(C)$ by setting

$$g \sim_n h \quad \Leftrightarrow \quad g \cdot x = h \cdot x \quad \text{for all } x \in \mathrm{M}_n(C). \tag{5.79}$$

Definition 5.32. *Let* $\mathcal{G}$ *be a presheaf on* $\mathsf{Comm}^{\mathrm{op}}$*, equipped with an action of* PGL_s*. Then we say that the action is* continuous *if for every finitely generated commutative algebra* C *and* $x \in \mathcal{G}(C)$*, there is some* $n \in \mathbb{N}_+$ *such that*

$$g \cdot x|_D = x|_D \tag{5.80}$$

for every morphism $C \to D$ *and* $g \in \mathrm{PGL}(D)$ *with* $g \sim_n 1$.

It is well-known that PGL_n for $n \in \mathbb{N}_+$ is representable by an affine scheme, so it is a sheaf for the canonical Grothendieck topology. Similarly, let J be a subcanonical topology, such that every J-cover has a finite subcover. Then we claim that PGL_s is a J-sheaf. It is enough to check the sheaf condition on a finite cover $\{C \to C_i\}_{i=1}^k$. Take elements $g_i \in \mathrm{PGL}_s(C_i)$ agreeing on intersections. We can assume that each $g_i \in \mathrm{PGL}_n(C_i)$ for some common $n \in \mathbb{N}_+$. Now there is a unique glued element $g \in \mathrm{PGL}_n(C) \subseteq \mathrm{PGL}_s(C)$. We can prove in an analogous way that M_s from (5.78) is a J-sheaf. Moreover, it is easy to see that the action of PGL_s on M_s is continuous (as in Definition 5.32).

More generally, we can take any finitely generated noncommutative algebra R over $\mathbb{C}$, and look at the functor

$$\mathsf{Comm} \to \mathsf{Sets} \qquad D \mapsto \mathsf{Alg}(R, \mathrm{M}_n(D)), \tag{5.81}$$

sending a commutative ring D to the set of algebra morphisms $R \to \mathrm{M}_n(D)$. It is well-known that this functor is representable by an affine scheme $\mathrm{rep}_n R$. The standard embeddings $\rho_{n,m} : \mathrm{M}_n(D) \to \mathrm{M}_m(D)$ induce a morphism of schemes $\mathrm{rep}_n R \to \mathrm{rep}_m R$. So we can again define a functor

$$(\mathrm{rep}_s R)(D) = \bigcup_{n|s} (\mathrm{rep}_n R)(D) \tag{5.82}$$

and with the same proof as for PGL_s we can see that this is a J-sheaf (under the same conditions: J subcanonical and every J-cover has a finite subcover).

We would like to define a *sheaf of trace preserving representations* $\mathrm{trep}_s R$ as well, for R an algebra with trace. For R an algebra, recall that a *trace* on R is a $\mathbb{C}$-linear function $\mathrm{tr} : R \to R$ such that for all $a, b \in R$

(a) (Maps into center) $\mathrm{tr}(a)b = b\,\mathrm{tr}(a)$;
(b) (Necklace property) $\mathrm{tr}(ab) = \mathrm{tr}(ba)$;
(c) (Linear with respect to traces) $\mathrm{tr}(\mathrm{tr}(a)b) = \mathrm{tr}(a)\mathrm{tr}(b)$.

Each Azumaya algebra A can be equipped with a trace. For A Azumaya of constant degree n and center C, we can take an étale cover $C \to D$ such that $A \otimes_C D \cong \mathrm{M}_n(D)$. Then the trace of $a \in A$ can be defined as the trace of the corresponding matrix in $\mathrm{M}_n(D)$. In particular, $\mathrm{tr}(1) = n$. For more about algebras with trace, we refer to Le Bruyn [LB08].

Now let $\varphi : A \to B$ be a center-preserving morphism, with A of constant degree n and B of constant degree nk. Then it is easy to see that

$$\varphi(\mathrm{tr}(a)) = \frac{1}{k}\mathrm{tr}(\varphi(a)), \tag{5.83}$$

so φ is trace-preserving if and only if A and B are of the same degree. To avoid this, we define the *normalized trace* of an Azumaya algebra A as

$$\mathrm{tr}'(a) = \frac{1}{n}\mathrm{tr}(a) \tag{5.84}$$

for all $a \in A$, with n the degree of A. In particular, $\mathrm{tr}'(1) = 1$. If A does not have constant degree, then the degree is at least locally constant, so we can define the normalized trace locally. For the normalized trace, morphisms between Azumaya algebras are trace-preserving if and only if they are center-preserving. Moreover, there is an obvious normalized trace on $\mathrm{M}_s(C)$ (well-known in the context of C^*-algebras).

If we work with normalized traces, it makes sense to look at the subsheaf

$$\mathrm{trep}_s(R) \subseteq \mathrm{rep}_s(R) \tag{5.85}$$

with as sections over C the morphisms $R \to \mathrm{M}_s(C)$ that are trace-preserving. In particular, for A an Azumaya algebra, the sheaf $\mathrm{trep}_s(A)$ is given by the center-preserving morphisms $A \to \mathrm{M}_s(C)$ (these are all embeddings).

Proposition 5.33. *Take a supernatural number $s \in \mathbb{S}$. Let A be a finitely generated Azumaya algebra of degree $n \mid s$ over its center C. Then* $\mathrm{trep}_s A$ *is an* étale PGL_s-homogeneous space *over* $\mathrm{Spec}(C)$*, in the sense that*

(a) ($\exists$ section locally) *we can find an étale covering $\{f_i : C \to C_i\}_{i \in I}$ and morphisms $\varphi_i : A \to \mathrm{M}_s(C_i)$ extending f_i, for all $i \in I$;*
(b) (sections in same orbit locally) *for any $f : C \to E$ and $\varphi, \psi : A \to \mathrm{M}_s(E)$ extending f, we can find an étale covering $\{f_i : E \to E_i\}_{i \in I}$ and automorphisms $g_i \in \mathrm{PGL}_s(E_i)$, such that $g_i \cdot \varphi|_{E_i} = \psi|_{E_i}$.*

$$\begin{array}{ccc} & \overset{\varphi|_{E_i}}{\nearrow} & \mathrm{M}_s(E_i) \\ A & & \downarrow g_i \\ & \underset{\psi|_{E_i}}{\searrow} & \mathrm{M}_s(E_i) \end{array} \tag{5.86}$$

In the case that s is completely infinite, $\mathrm{trep}_s(A)$ *is even a* Zariski PGL_s-homogeneous space*, i.e. the coverings can be Zariski coverings in the two conditions above.*

Proof. (a) Just take an étale covering $\{C \to C_i\}_{i\in I}$ trivializing A, i.e. such that $A \otimes_C C_i \cong \mathrm{M}_n(C_i)$. In the case that s is completely infinite, we can already find an embedding $A \to \mathrm{M}_s(C)$, using the result in Bass–Roy [BR67, Proposition 6.1] and the remark afterwards. So the trivial covering suffices.

(b) Take a natural number $m \mid s$ such that $\varphi(A)$ and $\psi(A)$ are both contained in $\mathrm{M}_m(E)$. Use the Double Centralizer Theorem to write

$$\varphi(A) \otimes_E B \cong \mathrm{M}_m(E) \cong \psi(A) \otimes_E B'. \tag{5.87}$$

Because $\varphi(A)$ and $\psi(A)$ are isomorphic, B and B' are in the same Brauer class. Take a Zariski covering $\{E \to E_i\}_{i\in I}$ such that $B \otimes_E E_i \cong B' \otimes_E E_i$. Now we get $g_i \in \mathrm{PGL}_m(E_i)$ by tensoring an isomorphism $\varphi(A) \to \psi(A)$ with an isomorphism $B \otimes_E E_i \to B' \otimes_E E_i$.

□

If A is an Azumaya algebra of degree $n \nmid s$, then $\mathrm{trep}_s(A)$ is the empty sheaf.

Theorem 5.34. *Take a supernatural number $s \in \mathbb{S}$. Take a subcanonical Grothendieck topology J such that every J-cover admits a finite subcover, and such that $J_s = J_{(K_s)}$ is a trivializing combined Grothendieck topology. Then there is an equivalence of toposes*

$$\mathsf{Sh}(\mathsf{Azu}^{\mathrm{op}}, J_s) \simeq \mathrm{PGL}_s - \mathsf{Sh}(\mathsf{Comm}^{\mathrm{op}}, J). \tag{5.88}$$

Here the right hand side is the topos of J-sheaves equipped with a continuous action of PGL_s *(as in Definition 5.32).*

Proof. Consider the functors

$$\mathsf{Sh}(\mathsf{Azu}^{\mathrm{op}}, J_s) \underset{\sharp}{\overset{\flat}{\rightleftarrows}} \mathrm{PGL}_s - \mathsf{Sh}(\mathsf{Comm}^{\mathrm{op}}, J) \tag{5.89}$$

defined by

$$\mathcal{F}^\flat(C) = \varinjlim_{n\mid s} \mathcal{F}(\mathrm{M}_n(C)) \tag{5.90}$$

$$\mathcal{G}^\sharp(A) = \mathrm{Hom}_{\mathrm{PGL}_s}(\mathrm{trep}_s A, \mathcal{G}). \tag{5.91}$$

Here the action of $g \in \mathrm{PGL}_n(C)$ on $\mathcal{F}(\mathrm{M}_n(C))$ is induced by the action on $\mathrm{M}_n(C)$, for all $n \mid s$. It is clear that the action is continuous (as in Definition 5.32).

<u>$\mathcal{F}^\flat$ is a sheaf.</u> Analogously to how we proved that PGL_s is a J-sheaf (after Definition 5.32), we can show that $\mathcal{F}^\flat$ is a J-sheaf whenever $\mathcal{F}$ is a J_s-sheaf. Here we need the assumption that every J-cover admits a finite subcover.

<u>$\mathcal{G}^\sharp$ is a sheaf.</u> For $\{\varphi_i : A \to A_i\}_{i\in I}$ a J_s-covering sieve, the sheaf morphism

$$\bigsqcup_{i\in I} \mathrm{trep}_s(A_i) \longrightarrow \mathrm{trep}_s A \tag{5.92}$$

is an epimorphism, because it is a surjection on stalks. This implies that if $\mathcal{G}$ is a sheaf, then $\mathcal{G}^\sharp$ is a separated presheaf. To show that it is a sheaf, take a matching family of sections

$$s_i \in \mathcal{G}^\sharp(A_i), \qquad i \in I \tag{5.93}$$

corresponding to sheaf morphisms

$$\psi_i : \mathrm{trep}_s A_i \longrightarrow \mathcal{G}. \tag{5.94}$$

The fact that $(s_i)_{i\in I}$ is a matching family of sections, translates as follows to a condition on the ψ_i's: if there is a commutative diagram

$$\begin{array}{ccccc} & & A_i & & \\ & \overset{\varphi_i}{\nearrow} & & \searrow & \\ A & & & & B \\ & \underset{\varphi_j}{\searrow} & & \nearrow & \\ & & A_j & & \end{array} \tag{5.95}$$

in Azu, with $i, j \in I$ and B arbitrary, then the corresponding diagram

$$\begin{array}{ccccc} & & \mathrm{trep}_s A_i & & \\ & \nearrow & & \overset{\psi_i}{\searrow} & \\ \mathrm{trep}_s B & & & & \mathcal{G} \\ & \searrow & & \underset{\psi_j}{\nearrow} & \\ & & \mathrm{trep}_s A_j & & \end{array} \tag{5.96}$$

commutes as well. In particular, take a diagram

$$\begin{array}{ccccc} & & A_i & & \\ & \overset{\varphi_i}{\nearrow} & & \overset{g}{\searrow} & \\ A & & & & A_i \\ & \underset{\varphi_i}{\searrow} & & \underset{\mathrm{id}}{\nearrow} & \\ & & A_i & & \end{array} \tag{5.97}$$

with g an automorphism of A_i fixing A (or more precisely, fixing $\varphi_i(A)$). Then it follows that

$$\psi_i(g \cdot y) = \psi_i(y) \quad \text{for every section } y \text{ of the sheaf } \mathrm{trep}_s A_i. \tag{5.98}$$

We now claim there is a unique

$$\psi : \mathrm{trep}_s A \longrightarrow \mathcal{G} \tag{5.99}$$

extending ψ_i for every $i \in I$. Take a morphism $\mathrm{Spec}(E) \to \mathrm{trep}_s A$ for some commutative ring E, given by an embedding

$$x : A \to \mathrm{M}_s(E). \tag{5.100}$$

We can take a J-cover $\{E \to E_m\}_{m \in M}$, and some $\hat{m} \in I$ for every $m \in M$, and a morphism $x_{\hat{m}}$, such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{x|_{E_m}} & \mathrm{M}_s(E_m) \\ {\scriptstyle \varphi_{\hat{m}}}\downarrow & \nearrow_{x_{\hat{m}}} & \\ A_{\hat{m}} & & \end{array} \tag{5.101}$$

commutes. Now define $\psi(x|_{E_m}) = \psi_{\hat{m}}(x_{\hat{m}})$. The image $\psi_{\hat{m}}(x_{\hat{m}})$ does not depend on the choice of $x_{\hat{m}}$, because for another choice $x'_{\hat{m}}$, we can (J-locally) find an automorphism $g \in \mathrm{PGL}_s(E_m)$ such that

$$x'_{\hat{m}} = g \cdot x_{\hat{m}} \tag{5.102}$$

(see Proposition 5.33). This g leaves A invariant, so we get

$$\psi_{\hat{m}}(x'_{\hat{m}}) = \psi_{\hat{m}}(g \cdot x_{\hat{m}}) = \psi_{\hat{m}}(x_{\hat{m}}). \tag{5.103}$$

So, locally, $\psi(x)$ is uniquely determined. In particular it agrees on intersections, so it is defined globally. This shows that $\mathcal{G}^\sharp$ is a sheaf.

$\underline{1 \simeq \sharp \circ \flat}$. Let $\mathcal{F}$ be a J_s-sheaf on $\mathsf{Azu}^{\mathrm{op}}$. By Yoneda Lemma, there is a natural bijective correspondence between elements $x \in \mathcal{F}(A)$ and presheaf morphisms $\mathbf{y}A \to \mathcal{F}$, or equivalently $(\mathbf{y}A)_{\downarrow s} \to \mathcal{F}$. Applying $\flat$ gives a morphism $\mathrm{trep}_s A \to \mathcal{F}^\flat$. This procedure yields a natural transformation

$$\eta : \mathcal{F} \to (\mathcal{F}^\flat)^\sharp. \tag{5.104}$$

We claim that this is an isomorphism. On a matrix algebra $\mathrm{M}_n(C)$ with C a commutative ring and $n \mid s$, the natural transformation $\eta : \mathcal{F} \to (\mathcal{F}^\flat)^\sharp$ is given by

$$\begin{array}{ccc} \mathcal{F}(\mathrm{M}_n(C)) & \xrightarrow{\eta} & \mathrm{Hom}_{\mathrm{PGL}_s}(\mathrm{trep}_s \mathrm{M}_n(C), \mathcal{F}^\flat) \\ y & \mapsto & (\varphi_y : \rho \mapsto \rho^* y). \end{array}$$

For the standard embedding $\rho_{n,s} : \mathrm{M}_n(C) \to \mathrm{M}_s(C)$, we get $\varphi_y(\rho_{n,s}) = \rho^*_{n,s}(y)$. Note that the restriction map $\rho^*_{n,s}$ on $\mathcal{F}$ is injective, because $\mathcal{F}$ is a J_s-sheaf, and $n \mid s$. So we can recover y from φ_y, in other words η is injective.

To show surjectivity, take $\varphi : \mathrm{trep}_s \mathrm{M}_n(C) \longrightarrow \mathcal{F}^\flat$ an arbitrary PGL_s-equivariant morphism. Set $y = \varphi(\rho_{n,s}) \in \varinjlim_{n|k} \mathcal{F}(\mathrm{M}_k(C))$. Because φ is equivariant, y is invariant under all $g \in \mathrm{PGL}_s(C)$ that leave $\rho_{n,s}$ invariant. So we can interpret y as an element of $\mathcal{F}(\mathrm{M}_n(C))$. It remains to show that $\varphi = \varphi_y$. Take an arbitrary $\rho : \mathrm{M}_n(C) \to \mathrm{M}_s(E)$ with E some finitely generated commutative ring. Let $f : C \to E$ be the restriction of ρ to C. Then $\rho_{n,s} \otimes f : \mathrm{M}_n(C) \to \mathrm{M}_s(E)$ is a center-preserving morphism with the same domain and codomain as ρ. Now by Proposition 5.33 we can find a Zariski covering $\{h_i : E \to E_i\}$ and automorphisms $g_i \in \mathrm{PGL}_s(E_i)$ such that

$$\rho|_{E_i} = g_i \cdot (\rho_{n,s} \otimes f_i) \tag{5.105}$$

with $f_i = h_i \circ f$. Now we can compute

$$\begin{aligned} \varphi(\rho|_{E_i}) &= \varphi(g_i \cdot (\rho_{n,s} \otimes f_i)) = g_i \cdot f_i^* \varphi(\rho_{n,s}) \\ &= g_i \cdot f_i^* y = (g_i \cdot (\rho_{n,s} \otimes f_i))^* y \\ &= (\rho|_{E_i})^* y = \varphi_y(\rho|_{E_i}). \end{aligned}$$

Here the third equality is the most difficult one, it follows from the definition of the action of PGL_s. The above shows that $\varphi = \varphi_y$, so η is surjective.

We have now proved that η is an isomorphism on matrix algebras. From this it easily follows that it is an isomorphism on all Azumaya algebras, because J_s is assumed to be trivializing.

$\underline{\flat \circ \sharp \simeq 1.}$ For $\mathcal{G}$ a PGL_s-sheaf, consider the natural transformation

$$\varepsilon : (\mathcal{G}^\sharp)^\flat \to \mathcal{G} \tag{5.106}$$

given by $\varepsilon(\varphi) = \varphi(\rho_{n,s})$, for

$$\varphi \in (\mathcal{G}^\sharp)^\flat(C) = \varinjlim_{n|s} \mathrm{Hom}_{\mathrm{PGL}_s}(\mathrm{trep}_s\, \mathrm{M}_n(C), \mathcal{G}). \tag{5.107}$$

It is easy to see that $\varphi(\rho_{n,s})$ does not depend on n, so ε is well-defined. For $y \in \mathcal{G}(C)$, we have to show that there is a unique φ_y such that $y = \varepsilon(\varphi_y) = \varphi_y(\rho_{n,s})$. Here $n \in \mathbb{N}_+$ is a natural number such that

$$g \cdot y|_E = y|_E \tag{5.108}$$

for all morphisms $C \to E$ and all elements $g \in \mathrm{PGL}_s(E)$ such that $g \sim_n 1$ (see Definition 5.32). We define φ_y as follows. Let $\rho : \mathrm{M}_n(C) \to \mathrm{M}_s(E)$ be a center-preserving morphism with E some finitely generated commutative ring, and let $f : C \to E$ be the restriction of ρ to C. Then by Proposition 5.33, we can take a Zariski cover $\{h_i : E \to E_i\}_{i \in I}$ and automorphisms $g_i \in \mathrm{PGL}_s(E_i)$ such that

$$\rho|_{E_i} = g_i \cdot (\rho_{n,s} \otimes f_i) \tag{5.109}$$

with $f_i = h_i \circ f$. We partially define φ_y as

$$\varphi_y(\rho|_{E_i}) = g_i \cdot f_i^* y. \tag{5.110}$$

This does not depend on the choice of g_i. Indeed, if g_i' is another choice, then $g_i' g_i^{-1} \sim_n 1$ by definition, so $g_i \cdot f_i^* y = g_i' \cdot f_i^* y$. In particular, $\varphi_y(\rho|_{E_i})$ and $\varphi_y(\rho|_{E_j})$ agree on the intersection, for $i \neq j$. So we can define φ_y globally. Moreover, because φ_y has to be equivariant, this is the unique possibility. □

Chapter 6

What is a noncommutative topos?

In [Con16], Connes claims that the Arithmetic Site (and the Scaling Site) are "[...] only the semiclassical shadows of a more mysterious structure underlying the compactification of Spec($\mathbb{Z}$) [...]". Additional evidence is given in [LB16], where Le Bruyn constructs a noncommutative topology that has the space of points of the Arithmetic Site (with the so-called sieve topology) as semiclassical shadow. In order to investigate the connection between the topos-theoretic approach on one hand and the noncommutative topology approach on the other hand, we will in this chapter introduce noncommutative toposes, as generalizations of toposes (similar to how noncommutative frames generalize frames in Cvetko-Vah [CV19]).

The set $\Omega = \mathcal{O}(X)$ of all open sets of a topological space X is a complete Heyting algebra: it is partially ordered under inclusion, the join $\vee$ and meet $\wedge$ operations are resp. union and intersection of opens, the implication operator $U \to V$ is defined to be the largest open set W satisfying $W \cap U \subseteq V$, and it has a unique bottom element $0 = \emptyset$ and top element $1 = X$, see for example Mac Lane–Moerdijk [MLM94, I.8].

Let $\mathcal{F}$ be a sheaf of sets over the constructible topology on X, that is the topology generated by all open *and* all closed subsets of X.[1] For every open set U in X we consider $\{(U, s) \mid s \in \mathcal{F}(U)\}$. The set H of all such possible (U, s) is partially ordered under $(U, s) \leq (V, t)$ if and only if $U \subseteq V$ and $t|U = s$. Fix a distinguished global section $g \in \mathcal{F}(X)$. We now define noncommutative operations of H as follows:

- $(U, s) \wedge (V, t) = (U \cap V, s|U \cap V)$,
- $(U, s) \vee (V, t) = (U \cup V, t \cup s|U - V)$,
- $(U, s) \to (V, t) = (U \to V, t \cup s|(U \to V) - V)$.

H still has a unique bottom element corresponding to $0 = \emptyset$, but now has a family $\{(X, t) \mid t \in \mathcal{F}(X)\}$ of top elements, and observe that the downset $\downarrow(X, t)$ for each of them is isomorphic to the Heyting algebra Ω, and if we consider

[1] For X the dcpo of filters on a poset P, as in Chapter 2, the constructible topology is the same as the strong topology from Definition 2.5.

Green's equivalence relation $\mathcal{D}$

$$(U,s)\ \mathcal{D}\ (V,t) \quad \text{if and only if} \quad \begin{cases} (U,s)\wedge(V,t)\wedge(U,s)=(U,s) \\ (V,t)\wedge(U,s)\wedge(V,t)=(V,t) \end{cases} \tag{6.1}$$

then the equivalence classes $H/\mathcal{D}$ with the induced structures are isomorphic to Ω as Heyting algebras. H is an example of a noncommutative complete Heyting algebra as introduced and studied in Cvetko-Vah [CV19]. We can view H as the set of opens of a noncommutative topological space with commutative shadow X.

In this Chapter we aim to define, in a similar way, noncommutative counterparts of toposes $\mathsf{Sh}(\mathcal{C},J)$ of sheaves of sets with respect to a Grothendieck topology J on a small category $\mathcal{C}$. Fred Van Oystaeyen suggested in his book 'Virtual topology and functor geometry' a possible approach:

> "One easily finds that the first main problem is to circumvent the notion of subobject classifier. An approach may be to allow a *family* of 'subobject classifiers' defined in a suitable way." [VO08, p. 44]

Let $\mathsf{PSh}(\mathcal{C})$ be the topos of presheaves on $\mathcal{C}$. Recall from Mac Lane–Moerdijk [MLM94, III.7] that the natural transformation $true : \mathbf{1} \longrightarrow \mathbf{\Omega}$ is the subobject classifier of $\mathsf{PSh}(\mathcal{C})$, where for every object C of $\mathcal{C}$ we take $\mathbf{\Omega}(C)$ to be the set of all sieves on C and where the global section $true$ picks out the unique maximal sieve $\mathbf{y}(C)$ of all morphisms with codomain C. Each $\mathbf{\Omega}(C)$ is a complete Heyting algebra, that is, $\mathbf{\Omega}$ is a presheaf of complete Heyting algebras on $\mathcal{C}$. We will define a *noncommutative subobject classifier* $\mathbf{H}$ to be a presheaf of noncommutative complete Heyting algebras making the diagram below commute

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad \mathbf{\Omega} \quad} & \mathsf{cHA} \\ & {\scriptstyle \mathbf{H}}\searrow \quad \nearrow{\scriptstyle ./\mathcal{D}} & \\ & \mathsf{ncHA} & \end{array} \tag{6.2}$$

where $./\mathcal{D} : \mathsf{ncHA} \longrightarrow \mathsf{cHA}$ is the covariant functor sending a noncommutative complete Heyting algebra H to its commutative shadow $H/\mathcal{D}$. Note that $\mathbf{H}$ has a subobject $t_{\mathbf{H}} : \mathbf{T} \longrightarrow \mathbf{H}$ where $\mathbf{T}$ is the presheaf of top elements of $\mathbf{H}$. We will often recite these two mantras:

(a) Occurrences of the terminal object $\mathbf{1}$ and $\mathbf{\Omega}$ in classical definitions should be replaced by the presheaves $\mathbf{T}$ and $\mathbf{H}$.

(b) All noncommutative structures will determine *families* of classical structures, parametrized by the global sections of $\mathbf{T}$.

Let us illustrate this in the definition of the *noncommutative Heyting algebra* $\mathbf{Sub_H}(\mathbf{P})$ generalizing the classical Heyting algebra of subobjects $\mathbf{Sub}(\mathbf{P})$ of $\mathbf{P}$ in $\mathsf{PSh}(\mathcal{C})$. Subobjects of $\mathbf{P}$ are in one-to-one correspondence with natural transformations $N : \mathbf{P} \longrightarrow \mathbf{\Omega}$ via the pullback diagram on the left below

$$\begin{array}{ccc} \mathbf{Q} & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow{\scriptstyle true} \\ \mathbf{P} & \xrightarrow[N]{} & \mathbf{\Omega} \end{array} \qquad\qquad \begin{array}{ccc} \mathbf{Q} & \longrightarrow & \mathbf{T} \\ \downarrow & & \downarrow{\scriptstyle t_{\mathbf{H}}} \\ \mathbf{P} & \xrightarrow[N]{} & \mathbf{H} \end{array}$$

Similarly, elements of $\mathbf{Sub}_{\mathbf{H}}(\mathbf{P})$ will be pairs $(\mathbf{Q}, N)$ where $N : \mathbf{P} \longrightarrow \mathbf{H}$ is a natural transformation and $\mathbf{Q}$ is the pullback subobject of the diagram on the right above. Because $\mathbf{H}$ is a presheaf of noncommutative Heyting algebras we have that if N and N' are natural transformations from $\mathbf{P}$ to $\mathbf{H}$ then so are $N \wedge N'$, $N \vee N'$ and $N \to N'$ as defined in Lemma 6.3. This then allows us to define operations on $\mathbf{Sub}_{\mathbf{H}}(\mathbf{P})$

$$\begin{cases} (\mathbf{Q}, N) \wedge (\mathbf{Q}', N') = (\mathbf{Q} \wedge \mathbf{Q}', N \wedge N') \\ (\mathbf{Q}, N) \vee (\mathbf{Q}', N') = (\mathbf{Q} \vee \mathbf{Q}', N \vee N') \\ (\mathbf{Q}, N) \to (\mathbf{Q}', N') = (\mathbf{Q} \to \mathbf{Q}', N \to N') \end{cases} \tag{6.3}$$

where we have the pull-back diagrams

$$\begin{array}{ccc} \mathbf{Q} \wedge \mathbf{Q}' & \longrightarrow & \mathbf{T} \\ \downarrow & & \downarrow t_{\mathbf{H}} \\ \mathbf{P} & \xrightarrow[N \wedge N']{} & \mathbf{H} \end{array} \qquad \begin{array}{ccc} \mathbf{Q} \vee \mathbf{Q}' & \longrightarrow & \mathbf{T} \\ \downarrow & & \downarrow t_{\mathbf{H}} \\ \mathbf{P} & \xrightarrow[N \vee N']{} & \mathbf{H} \end{array} \qquad \begin{array}{ccc} \mathbf{Q} \to \mathbf{Q}' & \longrightarrow & \mathbf{T} \\ \downarrow & & \downarrow t_{\mathbf{H}} \\ \mathbf{P} & \xrightarrow[N \to N']{} & \mathbf{H} \end{array} \tag{6.4}$$

defining a noncommutative Heyting algebra structure. Let $\Gamma(\mathbf{T})$ be the set of global sections $g : \mathbf{1} \longrightarrow \mathbf{T}$ of the presheaf of top elements $\mathbf{T}$, then there is a morphism

$$\mathbf{Sub}_{\mathbf{H}}(\mathbf{P}) \longrightarrow \prod_{g \in \Gamma(\mathbf{T})} \mathbf{Sub}(\mathbf{P}) \qquad (\mathbf{Q}, N) \mapsto (\mathbf{Q}_g)_{g \in \Gamma(\mathbf{T})} \tag{6.5}$$

with $\mathbf{Q}_g$ determined by the diagram below

$$\begin{array}{ccccc} \mathbf{Q}_g & \longrightarrow & \mathbf{1} & & \\ \downarrow & & \downarrow g & \searrow^{id} & \\ \mathbf{Q} & \xrightarrow[N]{} & \mathbf{T} & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow t_{\mathbf{H}} & & \downarrow true \\ \mathbf{P} & \xrightarrow[N]{} & \mathbf{H} & \xrightarrow[./\mathcal{D}]{} & \mathbf{\Omega} \end{array} \tag{6.6}$$

Having defined noncommutative subobject classifiers $\mathbf{H}$, we approach defining noncommutative Grothendieck topologies via generalizing Lawvere–Tierney topologies on $\mathsf{PSh}(\mathcal{C})$, see for example Mac Lane–Moerdijk [MLM94, V.1]. A *noncommutative Lawvere topology* will then be a natural transformation $j_{\mathbf{H}} : \mathbf{H} \longrightarrow \mathbf{H}$ satisfying

(NLT1) $j_{\mathbf{H}} \circ t_{\mathbf{H}} = t_{\mathbf{H}}$,

(NLT2) $j_{\mathbf{H}} \circ j_{\mathbf{H}} = j_{\mathbf{H}}$,

$$\begin{array}{ccc} \mathbf{T} & \xrightarrow{t_{\mathbf{H}}} & \mathbf{H} \\ & {}_{t_{\mathbf{H}}}\searrow & \downarrow j_{\mathbf{H}} \\ & & \mathbf{H} \end{array} \qquad \begin{array}{ccc} \mathbf{H} & \xrightarrow{j_{\mathbf{H}}} & \mathbf{H} \\ & {}_{j_{\mathbf{H}}}\searrow & \downarrow j_{\mathbf{H}} \\ & & \mathbf{H} \end{array} \tag{6.7}$$

(NLT3) For every object C in $\mathcal{C}$, every top element $t \in \mathbf{T}(C)$ and all $x, y \in \downarrow t \subset \mathbf{T}(C)$ we have the condition

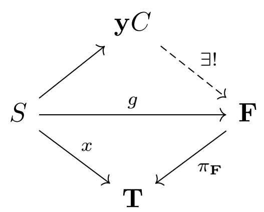
$$j_{\mathbf{H}}(C)(x \wedge y) = j_{\mathbf{H}}(C)(x) \wedge j_{\mathbf{H}}(C)(y). \tag{6.8}$$

Again, every global section $g : \mathbf{1} \longrightarrow \mathbf{T}$ determines a Lawvere–Tierney topology on $\mathsf{PSh}(\mathcal{C})$ via the restriction of $j_{\mathbf{H}}$ on $\downarrow g \simeq \mathbf{\Omega}$.

As $\mathcal{C}$ is a small category there is a one-to-one correspondence between Lawvere–Tierney topologies on $\mathsf{PSh}(\mathcal{C})$ and Grothendieck topologies on $\mathcal{C}$. Extending this, we have that a noncommutative Lawvere topology determines a *noncommutative Grothendieck topology* by associating to every object C the following collection of elements from $\mathbf{Sub}_{\mathbf{H}}(\mathbf{y}C)$:

$$J_{\mathbf{H}}(C) = \{(S, x) \in \mathbf{\Omega}(C) \times \mathbf{H}(C) \mid (S, x) \in \mathbf{Sub}_{\mathbf{H}}(\mathbf{y}C) \text{ and } j_{\mathbf{H}}(C)(x) \in \mathbf{T}(C)\} \tag{6.9}$$

This then allows us to define a presheaf $\mathbf{F}$ in the slice category $\mathsf{PSh}(\mathcal{C})/\mathbf{T}$ to be a *sheaf* for the noncommutative Grothendieck topology $J_{\mathbf{H}}$ if and only if for every object C of $\mathcal{C}$, every element $(S, x) \in J_{\mathbf{H}}(C)$, and every morphism g in $\mathsf{PSh}(\mathcal{C})/\mathbf{T}$



(6.10)

there is a unique morphism $\mathbf{y}C \longrightarrow \mathbf{F}$ in $\mathsf{PSh}(\mathcal{C})$. Here $S \xrightarrow{x} \mathbf{T}$ is the pull-back map induced by the natural transformation $x : \mathbf{y}C \longrightarrow \mathbf{H}$. The category of all such sheaves $\mathsf{Sh}(\mathcal{C}, J_{\mathbf{H}})$ is then called a *noncommutative topos*.

In the last section we present a large class of examples of noncommutative subobject classifiers and give an explicit example of a noncommutative topos which is *not* a Grothendieck topos, nor even an elementary topos.

6.1 Noncommutative Heyting algebras

In this section we will recall the main structural results on noncommutative (complete) Heyting algebras obtained in Cvetko-Vah [CV19].

Recall that a *bounded lattice* L is a set with two distinguished elements 0 and 1 and two binary operations $\vee$ and $\wedge$ which are both idempotent, associative and commutative and satisfy the identities

$$1 \wedge x = x, \qquad 0 \vee x = x, \tag{6.11}$$

$$x \wedge (y \vee x) = x = (x \wedge y) \vee x. \tag{6.12}$$

L is said to be *distributive* if we have the added identity

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z). \tag{6.13}$$

A *Heyting algebra* H is a bounded distributive lattice $(H, 0, 1, \vee, \wedge)$ which is also a partially ordered set under $\leq$ and has a binary operation $\to$ satisfying the following set of axioms

(H1) $(x \to x) = 1$,
(H2) $x \wedge (x \to y) = x \wedge y$,
(H3) $y \wedge (x \to y) = y$,

(H4) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

Equivalently, these axioms can be replaced by the following single axiom

(HA) $x \wedge y \leq z$ iff $x \leq y \to z$.

A Heyting algebra H is said to be *complete* if every subset $\{x_i : i \in I\}$ of H has a supremum $\bigvee_i x_i$ and an infimum $\bigwedge_i x_i$, satisfying the infinite distributive law $\bigvee_i (y \wedge x_i) = y \wedge \bigvee_i x_i$. With cHA we denote the category of all complete Heyting algebras with morphisms preserving arbitrary joins and meets (it is well-known that then the implication operator is preserved as well).

In Cvetko-Vah [CV19] noncommutative Heyting algebras were introduced and studied. A *skew lattice* is an algebra $(L, \wedge, \vee)$ where $\wedge$ and $\vee$ are idempotent and associative binary operations satisfying the identities

$$x \wedge (x \vee y) = x = x \vee (x \wedge y) \quad \text{and} \quad (x \wedge y) \vee y = y = (x \vee y) \wedge y \tag{6.14}$$

A skew lattice is *strongly distributive* if it satisfies the additional identities

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z) \quad \text{and} \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{6.15}$$

Green's equivalence relation $\mathcal{D}$ on a skew lattice is defined via $x \, \mathcal{D} \, y$ iff $x \wedge y \wedge x = x$ and $y \wedge x \wedge y = y$. We will denote the $\mathcal{D}$-equivalence class of $x \in L$ by $\mathcal{D}_x$. The set of equivalence classes $L/\mathcal{D}$ with the induced operations is a distributive lattice and if $L/\mathcal{D}$ has a maximal element 1 we call the corresponding $\mathcal{D}$-class in L the set of *top elements* and denote it with T.

A skew lattice has a natural partial order defined by $x \leq y$ iff $x \wedge y = x = y \wedge x$. With $\downarrow x$ we will denote the subset consisting of all $y \in L$ such that $y \leq x$. By a result of Leech [Lee92], $\downarrow x$ is a distributive lattice for any x in a strongly distributive skew lattice S. If S has a maximal element 1 then $S = \downarrow 1$, which implies that S is necessarily commutative. That is, we have to sacrifice a unique top element when passing to the noncommutative setting.

From Cvetko-Vah [CV19, §3] we recall that a *noncommutative Heyting algebra* is an algebra $(H, \wedge, \vee, 0, t)$ where $(H, \wedge, \vee, 0)$ is a strongly distributive lattice with bottom 0 and a top $\mathcal{D}$-class T, t is a distinguished element of T and $\to$ is a binary operation satisfying the following conditions

(NH1) $x \to y = (y \vee (t \wedge x \wedge t) \vee y) \to y$,
(NH2) $x \to x = x \vee t \vee x$,
(NH3) $x \wedge (x \to y) \wedge x = x \wedge y \wedge x$,
(NH4) $y \wedge (x \to y) = y$ and $(x \to y) \wedge y = y$,
(NH5) $x \to (t \wedge (y \wedge z) \wedge t) = (x \to (t \wedge y \wedge t)) \wedge (x \to (t \wedge z \wedge t))$.

The main structural result on noncommutative Heyting algebras, see Cvetko-Vah [CV19, Theorem 3.5], asserts that if $(H, \wedge, \vee, \to, 0, t)$ is a noncommutative Heyting algebra, then

(a) $(\downarrow t, \wedge, \vee, \to, 0, t)$ is a Heyting algebra with a unique top element t, isomorphic to $H/\mathcal{D}$;
(b) For any $t' \in T$ also $(\downarrow t', \wedge, \vee, \to, 0, t')$ is a Heyting algebra and the map

$$\varphi \; : \downarrow t \longrightarrow \downarrow t' \qquad x \mapsto t' \wedge x \wedge t' \tag{6.16}$$

is an isomorphism of Heyting algebras and for all $x \in \downarrow t$ we have $x \, \mathcal{D} \, \varphi(x)$.

From now on we will assume that the noncommutative Heyting algebra is *complete*, that is if all *commuting* subsets have supremums and infimums in their partial ordering, and they satisfy the infinite distributive laws

$$(\bigvee_i x_i) \wedge y = \bigvee_i (x_i \wedge y) \quad \text{and} \quad x \wedge (\bigvee_i y_i) = \bigvee_i (x \wedge y_i) \tag{6.17}$$

for all $x, y \in H$ and all commuting subsets $(x_i)_i$ and $(y_i)_i$.

With ncHA we denote the category with objects all complete noncommutative Heyting algebras and maps preserving $\leq$, $\wedge$, $\vee$, arbitrary joins and meets of commuting subsets, 0 and the distinguished top element t. By Cvetko-Vah [CV19, Theorem 4.5], the implication operator $\to$ is then preserved as well.

From Cvetko-Vah [CV19, Theorem 3.5.(iii)] we recall that Green's relation $\mathcal{D}$ is a congruence on a noncommutative Heyting algebra H and that the Heyting algebra $H/\mathcal{D}$ is its *maximal lattice image*, that is, every noncommutative Heyting algebra morphism $H \longrightarrow H_c$ to a (commutative) Heyting algebra H_c factors through the quotient $\pi_{\mathcal{D}} : H \twoheadrightarrow H/\mathcal{D}$. We can rephrase this as:

Lemma 6.1. *Green's relation $\mathcal{D}$ induces a covariant functor*

$$/\mathcal{D} \ : \ \mathsf{ncHA} \longrightarrow \mathsf{cHA} \qquad H \mapsto H/\mathcal{D} \tag{6.18}$$

and this functor is left adjoint to the inclusion $\mathsf{cHA} \longrightarrow \mathsf{ncHA}$.

6.2 Noncommutative subobject classifiers

Let $\mathcal{C}$ be a small category and $\mathbf{P}$ a presheaf on $\mathcal{C}$. We recall that subobjects of $\mathbf{P}$ correspond to natural transformations $N : \mathbf{P} \longrightarrow \mathbf{\Omega}$ to the subobject classifier $\mathbf{\Omega}$, which is a presheaf of complete Heyting algebras on $\mathcal{C}$.

Motivated by this, we will consider the set $(\mathbf{P}, \mathbf{H})$ of all natural transformations $N : \mathbf{P} \longrightarrow \mathbf{H}$ to a presheaf $\mathbf{H}$ of noncommutative complete Heyting algebras on $\mathcal{C}$ and equip this set with a noncommutative Heyting algebra structure.

Note that $\mathbf{\Omega}$ is a presheaf on $\mathcal{C}$. For a map $h : D \to C$ in $\mathcal{C}$, and a sieve $S \in \mathbf{\Omega}(C)$, the restriction h^*S is defined as the sieve

$$h^*S = h^{-1}S = \{g : g \circ h \in S\}. \tag{6.19}$$

As unions and intersections of sieves on C are again sieves on C, each $\mathbf{\Omega}(C)$ is a complete Heyting algebra with bottom element $0 = \emptyset$ and unique maximal element $1 = \mathbf{y}(C)$ the set of all morphisms with codomain C. Moreover, for any $h : D \longrightarrow C$ we have that the restriction morphism $h^* : \mathbf{\Omega}(C) \longrightarrow \mathbf{\Omega}(D)$ is a morphism of Heyting algebras. That is, we have a contravariant functor

$$\mathbf{\Omega} : \mathcal{C} \longrightarrow \mathsf{cHA} \tag{6.20}$$

to the category cHA of complete Heyting algebras. Assigning to each C the maximal element $1 = \mathbf{y}(C)$ defines a global section of $\mathbf{\Omega}$

$$true \ : \ \mathbf{1} \longrightarrow \mathbf{\Omega} \tag{6.21}$$

which is the subobject classifier in $\mathsf{PSh}(\mathcal{C})$, the topos of all presheaves of sets on $\mathcal{C}$, see Mac Lane–Moerdijk [MLM94, p. 37-39]. That is, for every presheaf

$\mathbf{P} \in \mathsf{PSh}(\mathcal{C})$ there is a natural one-to-one correspondence between natural transformations $N : \mathbf{P} \longrightarrow \mathbf{\Omega}$ and subobjects $\mathbf{Q}$ of $\mathbf{P}$ in $\mathsf{PSh}(\mathcal{C})$, given by the pullback diagram

$$\begin{array}{ccc} \mathbf{Q} & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow{\scriptstyle true} \\ \mathbf{P} & \underset{N}{\longrightarrow} & \mathbf{\Omega} \end{array} \tag{6.22}$$

With this in mind, let us start with a presheaf $\mathbf{H}$ of noncommutative complete Heyting algebras on $\mathcal{C}$, that is, a contravariant functor

$$\mathbf{H} \ : \ \mathcal{C} \longrightarrow \mathsf{ncHA} \tag{6.23}$$

Every morphism $D \xrightarrow{f} C$ in $\mathcal{C}$ induces a morphism of noncommutative complete Heyting algebras

$$H(f) : \mathbf{H}(C) \longrightarrow \mathbf{H}(D) \tag{6.24}$$

and, in particular, it induces a map on the sets of top elements of these noncommutative Heyting algebras

$$\mathbf{T}(f) \ : \ \mathbf{T}(C) = T(\mathbf{H}(C)) \longrightarrow T(\mathbf{H}(D)) = \mathbf{T}(D) \tag{6.25}$$

That is, taking for every object C in $\mathcal{C}$ the set of top elements $\mathbf{T}(C)$ of the noncommutative complete Heyting algebra $\mathbf{H}(C)$ is a presheaf of sets on $\mathcal{C}$, and the inclusions $\mathbf{T}(C) \subseteq \mathbf{H}(C)$ define a natural transformation

$$t_{\mathbf{H}} \ : \ \mathbf{T} \longrightarrow \mathbf{H}. \tag{6.26}$$

Lemma 6.2. *Let $\mathbf{P} \in \mathsf{PSh}(\mathcal{C})$ and let $N, N' : \mathbf{P} \longrightarrow \mathbf{H}$ be natural transformations. Then the maps*

$$\begin{cases} (N \wedge N')(C) : \mathbf{P}(C) \longrightarrow \mathbf{H}(C) & x \mapsto N(C)(x) \wedge N'(C)(x) \\ (N \vee N')(C) : \mathbf{P}(C) \longrightarrow \mathbf{H}(C) & x \mapsto N(C)(x) \vee N'(C)(x) \\ (N \rightarrow N')(C) : \mathbf{P}(C) \longrightarrow \mathbf{H}(C) & x \mapsto N(C)(x) \rightarrow N'(C)(x) \end{cases} \tag{6.27}$$

define natural transformation $N \wedge N', N \vee N', N \rightarrow N' : \mathbf{P} \longrightarrow \mathbf{H}$.

Proof. For every morphism $D \xrightarrow{f} C$ in $\mathcal{C}$ we have to verify that the diagram below is commutative

$$\begin{array}{ccc} \mathbf{P}(C) & \xrightarrow{(N \wedge N')(C)} & \mathbf{H}(C) \\ \downarrow{\scriptstyle \mathbf{P}(f)} & & \downarrow{\scriptstyle \mathbf{H}(f)} \\ \mathbf{P}(D) & \underset{(N \wedge N')(D)}{\longrightarrow} & \mathbf{H}(D) \end{array} \tag{6.28}$$

For every $x \in \mathbf{P}(C)$ we have that $\mathbf{H}(f)((N \wedge N')(C)(x)) =$

$$\mathbf{H}(f)(N(C)(x) \wedge N'(C)(x)) = \mathbf{H}(f)(N(C)(x)) \wedge \mathbf{H}(f)(N'(C)(x)) \tag{6.29}$$

where the last equality follows from $\mathbf{H}(f)$ being a morphism of noncommutative complete Heyting algebras. Because N and N' are natural transformations, we have the equalities

$$\mathbf{H}(f)(N(C)(x)) = N(D)(\mathbf{P}(f)(x)) \quad \text{and} \quad \mathbf{H}(f)(N'(C)(x)) = N'(D)(\mathbf{P}(f)(x))$$

and so the term above is equal to

$$N(D)(\mathbf{P}(f)(x)) \wedge N'(D)(\mathbf{P}(f)(x)) = (N \wedge N')(D)(\mathbf{P}(f)(x))$$

The proofs for $N \vee N'$ and $N \to N'$ proceed similarly. □

Every natural transformation $N : \mathbf{P} \longrightarrow \mathbf{H}$ determines a pair $(\mathbf{Q}, N)$ where $\mathbf{Q}$ is a subobject of $\mathbf{P}$ via the pullback diagram

$$\begin{array}{ccc} \mathbf{Q} & \longrightarrow & \mathbf{T} \\ \downarrow & & \downarrow t_{\mathbf{H}} \\ \mathbf{P} & \xrightarrow[N]{} & \mathbf{H} \end{array} \tag{6.30}$$

With $\mathbf{Sub_H}(\mathbf{P})$ we denote the set of all such pairs $(\mathbf{Q}, N)$ determined by a natural transformation $N : \mathbf{P} \longrightarrow \mathbf{H}$.

Lemma 6.3. *On the poset* $\mathbf{Sub_H}(\mathbf{P})$ *we can define operations*

$$\begin{cases} (\mathbf{Q}, N) \wedge (\mathbf{Q}', N') = (\mathbf{Q} \wedge \mathbf{Q}', N \wedge N') \\ (\mathbf{Q}, N) \vee (\mathbf{Q}', N') = (\mathbf{Q} \vee \mathbf{Q}', N \vee N') \\ (\mathbf{Q}, N) \to (\mathbf{Q}', N') = (\mathbf{Q} \to \mathbf{Q}', N \to N') \end{cases}$$

where we have the pull-back diagrams

$$\begin{array}{ccc} \mathbf{Q} \wedge \mathbf{Q}' & \longrightarrow & \mathbf{T} \\ \downarrow & & \downarrow t_{\mathbf{H}} \\ \mathbf{P} & \xrightarrow[N \wedge N']{} & \mathbf{H} \end{array} \qquad \begin{array}{ccc} \mathbf{Q} \vee \mathbf{Q}' & \longrightarrow & \mathbf{T} \\ \downarrow & & \downarrow t_{\mathbf{H}} \\ \mathbf{P} & \xrightarrow[N \vee N']{} & \mathbf{H} \end{array} \qquad \begin{array}{ccc} \mathbf{Q} \to \mathbf{Q}' & \longrightarrow & \mathbf{T} \\ \downarrow & & \downarrow t_{\mathbf{H}} \\ \mathbf{P} & \xrightarrow[N \to N']{} & \mathbf{H} \end{array}$$

These operations turn the set $\mathbf{Sub_H}(\mathbf{P})$ *into a noncommutative complete Heyting algebra with minimal element* $(\emptyset, N_0)$ *and distinguished top element* $(\mathbf{P}, N_d)$, *where the natural transformations* $N_0, N_d : \mathbf{P} \longrightarrow \mathbf{H}$ *are the compositions*

$$N_0 : \mathbf{P} \longrightarrow \mathbf{1} \xrightarrow{0} \mathbf{H} \quad \text{and} \quad N_d : \mathbf{P} \longrightarrow \mathbf{1} \xrightarrow{d} \mathbf{H}$$

with the left-most morphism the unique map to the terminal object $\mathbf{1}$ *and* d *the global section of* $\mathbf{H}$ *determined by the distinguished elements. The top elements are exactly the pairs* $(\mathbf{P}, N)$ *where* $N : \mathbf{P} \longrightarrow \mathbf{T}$ *is a natural transformation.*

Proof. Follows from the previous lemma and uniqueness of pullbacks. □

Definition 6.4. *A presheaf* $\mathbf{H}$ *of noncommutative complete Heyting algebras on* $\mathcal{C}$ *is said to be a* noncommutative subobject classifier *if* $\mathbf{H}/\mathcal{D} \simeq \mathbf{\Omega}$.

Lemma 6.5. *If* $\mathbf{H}$ *is a noncommutative subobject classifier, then for every presheaf* $\mathbf{P}$ *on* $\mathcal{C}$, *we have a surjective morphism of (noncommutative) complete Heyting algebras*

$$\mathbf{Sub_H}(\mathbf{P}) \twoheadrightarrow \mathbf{Sub}(\mathbf{P}) \tag{6.31}$$

Proof. The map is determined by sending a pair $(\mathbf{Q}, N)$ to $\mathbf{Q}$. Or, equivalently, by composing with the quotient map of noncommutative complete Heyting algebras dividing out Green's relation

$$\begin{array}{ccccc} \mathbf{Q} & \longrightarrow & \mathbf{T} & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow t_{\mathbf{H}} & & \downarrow \\ \mathbf{P} & \xrightarrow[N]{} & \mathbf{H} & \xrightarrow{./\mathcal{D}} & \mathbf{\Omega} \end{array} \tag{6.32}$$

Let $d : \mathbf{1} \longrightarrow \mathbf{H}$ be the global section corresponding to the distinguished top element, then the maps (of noncommutative complete Heyting algebras)

$$\mathbf{\Omega}(C) \xrightarrow{\simeq} \downarrow d(C)(1) \hookrightarrow \mathbf{H}(C) \tag{6.33}$$

determine a natural transformation $\mathbf{\Omega} \xrightarrow{i} \mathbf{H}$. If $\mathbf{Q}$ is the subobject of $\mathbf{P}$ corresponding to the natural transformation $N : \mathbf{P} \longrightarrow \mathbf{\Omega}$ then the composition $i \circ N$ is an element of $(\mathbf{P}, \mathbf{H})$ mapping to $\mathbf{Q}$. □

6.3 Noncommutative Grothendieck topologies

In this section we will introduce noncommutative Grothendieck topologies and their corresponding toposes of sheaves. We will first extend the notion of Lawvere–Tierney topologies, which are certain closure operations on $\mathbf{\Omega}$, to noncommutative subobject classifiers. As Lawvere–Tierney topologies coincide with Grothendieck topologies when the category $\mathcal{C}$ is small, we will then determine the corresponding noncommutative Grothendieck topologies and define sheaves over them.

A *Lawvere–Tierney topology* on $\mathsf{PSh}(\mathcal{C})$, see for example Mac Lane–Moerdijk [MLM94, V.§1], is a natural transformation $j : \mathbf{\Omega} \longrightarrow \mathbf{\Omega}$ satisfying the following three properties

(LT1) $j \circ \text{true} = \text{true}$;
(LT2) $j \circ j = j$;
(LT3) $j \circ \wedge = \wedge \circ (j \times j)$.

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{true} & \mathbf{\Omega} \\ & \searrow^{true} & \downarrow j \\ & & \mathbf{\Omega} \end{array} \qquad \begin{array}{ccc} \mathbf{\Omega} & \xrightarrow{j} & \mathbf{\Omega} \\ & \searrow^{j} & \downarrow j \\ & & \mathbf{\Omega} \end{array} \qquad \begin{array}{ccc} \mathbf{\Omega} \times \mathbf{\Omega} & \xrightarrow{\wedge} & \mathbf{\Omega} \\ {\scriptstyle j \times j} \downarrow & & \downarrow j \\ \mathbf{\Omega} \times \mathbf{\Omega} & \xrightarrow[\wedge]{} & \mathbf{\Omega} \end{array} \tag{6.34}$$

Motivated by this we define, for a noncommutative subobject classifier $\mathbf{H}$ with presheaf of top elements $t_{\mathbf{T}} : \mathbf{T} \longrightarrow \mathbf{H}$, a *noncommutative Lawvere topology* to be a natural transformation (of presheaves of sets)

$$j_{\mathbf{H}} : \mathbf{H} \longrightarrow \mathbf{H} \tag{6.35}$$

satisfying the properties

(NLT1) $j_{\mathbf{H}} \circ t_{\mathbf{H}} = t_{\mathbf{H}}$,
(NLT2) $j_{\mathbf{H}} \circ j_{\mathbf{H}} = j_{\mathbf{H}}$,

$$\begin{array}{ccc} \mathbf{T} & \xrightarrow{t_{\mathbf{H}}} & \mathbf{H} \\ & \searrow^{t_{\mathbf{H}}} & \downarrow j_{\mathbf{H}} \\ & & \mathbf{H} \end{array} \qquad \begin{array}{ccc} \mathbf{H} & \xrightarrow{j_{\mathbf{H}}} & \mathbf{H} \\ & \searrow^{j_{\mathbf{H}}} & \downarrow j_{\mathbf{H}} \\ & & \mathbf{H} \end{array} \tag{6.36}$$

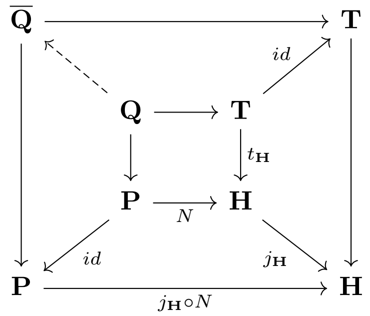
and where we replace the third commuting diagram by

(NLT3) For every object C in $\mathcal{C}$, every top-element $t \in \mathbf{T}(C)$ and all $x, y \in \downarrow t \subset \mathbf{H}(C)$ we have the condition

$$j_{\mathbf{H}}(C)(x \wedge y) = j_{\mathbf{H}}(C)(x) \wedge j_{\mathbf{H}}(C)(y). \tag{6.37}$$

Lemma 6.6. *A noncommutative Lawvere topology* $j_{\mathbf{H}} : \mathbf{H} \longrightarrow \mathbf{H}$ *induces for every presheaf* $\mathbf{P}$ *a closure operator on the noncommutative complete Heyting algebra* $\mathbf{Sub_H}(\mathbf{P})$.

Proof. Let $N : \mathbf{P} \longrightarrow \mathbf{H}$ be a natural transformation and consider the inner pullback square



(6.38)

then the composed morphism $j_{\mathbf{H}} \circ N$ gives the outer square, and hence determines an element in $\mathbf{Sub_H}(\mathbf{P})$

$$\overline{(\mathbf{Q}, N)} = (\overline{\mathbf{Q}}, j_{\mathbf{H}} \circ N) \tag{6.39}$$

The dashed morphism exists because the outer square is a pullback diagram, and hence we have $\mathbf{Q} \subseteq \overline{\mathbf{Q}}$ and therefore

$$(\mathbf{Q}, N) \leq \overline{(\mathbf{Q}, N)} \quad \text{and} \quad \overline{\overline{(\mathbf{Q}, N)}} = \overline{(\mathbf{Q}, N)} \tag{6.40}$$

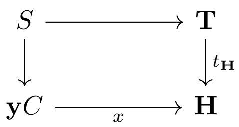
where the latter follows from $j_{\mathbf{H}} \circ j_{\mathbf{H}} = j_{\mathbf{H}}$. □

If $\mathcal{C}$ is a small category, Lawvere–Tierney topologies on $\mathsf{PSh}(\mathcal{C})$ are in one-to-one correspondence with Grothendieck topologies on $\mathcal{C}$, see for example Mac Lane and Moerdijk [MLM94, Theorem V.4.1]. One recovers the collection $J(C)$ from a Lawvere–Tierney topology j as the set of all sieves S on C such that $j(S) = \mathbf{y}(C)$ in $\mathbf{\Omega}(C)$.

Let us specify the construction of $\mathbf{Sub_H}(\mathbf{P})$ for the presheaf $\mathbf{P} = \mathbf{y}C$ determined by

$$\mathbf{y}C \;:\; \mathcal{C} \longrightarrow \mathbf{Sets} \qquad D \mapsto \mathcal{C}(D, C) \tag{6.41}$$

Note that the subobjects of $\mathbf{y}C$ are exactly the sieves S on C and that by Yoneda Lemma every natural transformation $N : \mathbf{y}C \longrightarrow \mathbf{H}$ determines (and is determined by) $x = N(C)(id_C) \in \mathbf{H}(C)$. Conversely, every element $x \in \mathbf{H}(C)$ determines the pull-back diagram



(6.42)

where S is the sieve on C specified by

$$S = \{D \xrightarrow{f} C \ : \ \mathbf{H}(f)(x) \in \mathbf{T}(D)\} \tag{6.43}$$

Observe that S is indeed a sieve as the maps $\mathbf{H}(g)$ for $E \xrightarrow{g} D$ induce a map on the top elements $\mathbf{T}(D) \longrightarrow \mathbf{T}(E)$. Therefore,

$$\mathbf{Sub_H}(\mathbf{y}C) = \{(S,x) \in \mathbf{\Omega}(C) \times \mathbf{H}(C) \mid S = \{D \xrightarrow{f} C \ : \ \mathbf{H}(f)(x) \in \mathbf{T}(D)\}\}$$

We have seen that $\mathbf{Sub_H}(\mathbf{y}C)$ is a noncommutative complete Heyting algebra, having as its set of top elements

$$T(\mathbf{Sub_H}(\mathbf{y}C)) = \{(\mathbf{y}(C), t) \mid t \in \mathbf{T}(C)\} \tag{6.44}$$

and with minimal element $(\emptyset, 0)$. If $j_{\mathbf{H}} : \mathbf{H} \longrightarrow \mathbf{H}$ is a noncommutative Lawvere topology, the corresponding closure operation on $\mathbf{Sub_H}(\mathbf{y}C)$ can be specified as

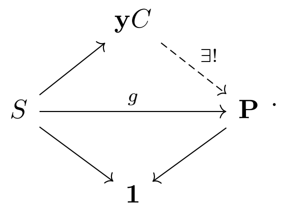
$$\overline{(S,x)} = (\overline{S}, j_{\mathbf{H}}(C)(x)) \quad \text{with} \quad \overline{S} = \{D \xrightarrow{f} C \ : \ \mathbf{T}(f)(j_{\mathbf{H}}(C)(x)) \in \mathbf{T}(D)\}$$

Motivated by the above correspondence between Lawvere–Tierney and Grothendieck topologies, we can now define:

Definition 6.7. *Let $j_{\mathbf{H}} : \mathbf{H} \longrightarrow \mathbf{H}$ be a noncommutative Lawvere topology, then the corresponding* noncommutative Grothendieck topology $J_{\mathbf{H}}$ *assigns to every object C of $\mathcal{C}$ the collection of elements from $\mathbf{Sub_H}(\mathbf{y}C)$*

$$J_{\mathbf{H}}(C) = \{(S,x) \in \mathbf{\Omega}(C) \times \mathbf{H}(C) \mid (S,x) \in \mathbf{Sub_H}(\mathbf{y}C) \text{ and } j_{\mathbf{H}}(C)(x) \in \mathbf{T}(C)\}$$

If J is a Grothendieck topology on $\mathcal{C}$ then a presheaf $\mathbf{P}$ of sets on $\mathcal{C}$ is called a sheaf for J if and only if for every object C of $\mathcal{C}$, every sieve $S \in J(C)$ (considered as a subobject of $\mathbf{y}C$) and every natural transformation $g : S \longrightarrow \mathbf{P}$, there is a unique natural transformation $\mathbf{y}C \longrightarrow \mathbf{P}$ making the diagram below commute



(6.45)

Clearly, the canonical bottom maps to the terminal object $\mathbf{1}$ are superfluous in the definition, but they may help to motivate the definition below.

Let $\mathbf{H}$ be a noncommutative subobject classifier with presheaf of top elements $\mathbf{T}$ and let $j_{\mathbf{H}} : \mathbf{H} \longrightarrow \mathbf{H}$ be a noncommutative Lawvere topology, then the corresponding noncommutative Grothendieck topology $J_{\mathbf{H}}$ assigns to every object C a collection $J_{\mathbf{H}}(C)$ of couples (S,x) where S is a subobject of $\mathbf{y}C$ and $x : S \longrightarrow \mathbf{T}$ is a natural transformation which is the restriction to S of a natural transformation $x : \mathbf{y}C \longrightarrow \mathbf{H}$ determined by $x \in \mathbf{H}(C)$.

So, instead of the canonical morphism $S \longrightarrow \mathbf{1}$ we have to consider certain morphisms $x : S \longrightarrow \mathbf{T}$. Therefore it makes sense to define the category of all presheaves with respect to the noncommutative Grothendieck topology $J_{\mathbf{H}}$ to be the slice category $\mathsf{PSh}(\mathcal{C})/\mathbf{T}$. That is, the objects are pairs $(\mathbf{F}, \pi_{\mathbf{F}})$ with $\mathbf{F} \in \mathsf{PSh}(\mathcal{C})$ and $\pi_{\mathbf{F}}$ a natural transformation $\mathbf{F} \longrightarrow \mathbf{T}$, and morphisms compatible natural transformations g

$$\begin{array}{c} \mathbf{F} \\ \downarrow{\scriptstyle \pi_{\mathbf{F}}} \\ \mathbf{T} \end{array} \qquad \begin{array}{ccc} \mathbf{F} & \xrightarrow{\quad g \quad} & \mathbf{G} \\ & {\scriptstyle \pi_{\mathbf{F}}} \searrow \quad \swarrow {\scriptstyle \pi_{\mathbf{G}}} & \\ & \mathbf{T} & \end{array} \quad . \tag{6.46}$$

Definition 6.8. *A presheaf* $(\mathbf{F}, \pi_{\mathbf{F}})$ *is a* sheaf *with respect to the noncommutative Grothendieck topology* $J_{\mathbf{H}}$ *if and only if for every object* C *of* $\mathcal{C}$*, every element* $(S, x) \in J_{\mathbf{H}}(C)$*, and every morphism* g *in* $\mathsf{PSh}(\mathcal{C})/\mathbf{T}$

$$\begin{array}{ccc} & \mathbf{y}C & \\ \nearrow & & \searrow {\scriptstyle \exists!} \\ S & \xrightarrow{\qquad g \qquad} & \mathbf{F} \\ {\scriptstyle x} \searrow & & \swarrow {\scriptstyle \pi_{\mathbf{F}}} \\ & \mathbf{T} & \end{array} \tag{6.47}$$

there is a unique morphism $\mathbf{y}C \longrightarrow \mathbf{F}$ *in* $\mathsf{PSh}(\mathcal{C})$*. Here* $S \xrightarrow{x} \mathbf{T}$ *is the pull-back map induced by the natural transformation* $x : \mathbf{y}C \longrightarrow \mathbf{H}$*.*

The noncommutative topos $\mathsf{Sh}(\mathcal{C}, J_{\mathbf{H}})$ *has as its objects all sheaves with respect to the noncommutative Grothendieck topology* $J_{\mathbf{H}}$ *and morphisms as in* $\mathsf{PSh}(\mathcal{C})/\mathbf{T}$*.*

6.4 A class of examples

In this section we will construct examples of noncommutative subobject classifiers and show that a noncommutative topos does not have to be an elementary topos.

First, we will construct complete noncommutative Heyting algebras. By a result of Cvetko-Vah [CV19] complete noncommutative Heyting algebras are exactly noncommutative frames (together with a distinguished element in the top $\mathcal{D}$-class), where a *noncommutative frame* is a strongly distributive, join complete skew lattice that satisfies the infinite distributive laws.

Let h be a (commutative) complete Heyting algebra. Since h is a distributive lattice, note that there is an embedding $i : h \longrightarrow \prod_{i \in I} \mathbf{2}$ for some index set I, where $\mathbf{2}$ is the two element lattice

$$\mathbf{2} = \begin{array}{c} 1 \\ | \\ 0 \end{array} \qquad \text{and define} \qquad \widehat{P} = \begin{array}{c} \cdots\cdots\; p \;\cdots\cdots \\ | \\ 0 \end{array} \tag{6.48}$$

to be the skew lattice on $\widehat{P} = \{0\} \cup P$, with a unique bottom element 0 and a set P of top elements, and operations are defined by:

$$x, y \in P : x \wedge y = x, \qquad x \vee y = y, \tag{6.49}$$

$$x \wedge 0 = 0 = 0 \wedge x, \qquad x \vee 0 = x = 0 \vee x. \tag{6.50}$$

Note that $\widehat{P}$ is a strongly distributive skew lattice and has two $\mathcal{D}$-classes: bottom class $\{0\}$ and top class P, whence $\widehat{P}/\mathcal{D} \simeq \mathbf{2}$.

Let H be the pullback (in **Sets**) of the following diagram:

$$\begin{array}{ccc} H & \longrightarrow & \prod_{i\in I} \widehat{P} \\ \big\downarrow & & \big\downarrow /\mathcal{D} \\ h & \xrightarrow{\ i\ } & \prod_{i\in I} \mathbf{2} \end{array} \tag{6.51}$$

Denoting by π_i the projection to the i-th factor we obtain a commutative diagram:

$$\begin{array}{ccc} H & \xrightarrow{\ \pi_i\ } & \widehat{P} \\ \big\downarrow & & \big\downarrow /\mathcal{D} \\ h & \longrightarrow & \mathbf{2} \end{array} \tag{6.52}$$

Lemma 6.9. *With notations as above, H becomes a noncommutative frame with bottom 0 and top $\mathcal{D}$-class $T(H) = \prod_{i\in I} P$ under the operations*

$$(x_i)_i \wedge (y_i)_i = (x_i \wedge y_i)_i \qquad \textit{and} \qquad (x_i)_i \vee (y_i)_i = (x_i \vee y_i)_i \tag{6.53}$$

where the bracketed operations are performed in the skew lattice $\widehat{P}$. In particular, $H/\mathcal{D} \simeq h$. If we fix a distinguished element $d \in H$ s.t. $\pi_i(d) \neq 0$ for all $i \in I$ then H is a complete noncommutative Heyting algebra.

Proof. First we observe that H is a strongly distributive skew lattice because it embeds into a power of $\widehat{P}$ and strongly distributive skew lattices form a variety. Note that elements $x, y \in H$ are $\mathcal{D}$-equivalent exactly when for all $i \in I$: ($\pi_i(x) = 0$ iff $\pi_i(y) = 0$). A commuting subset in H is of the form $\{x_j \,|\, j \in J\}$ such that $\pi_i(x_j) \neq 0$ together with $\pi_i(x_k) \neq 0$ implies $\pi_i(x_j) = \pi_i(x_k)$, for all $j, k \in J$ and all $i \in I$.

The skew lattice H is join complete because h is complete and the diagram 6.52 commutes. It remains to prove that H satisfies the infinite distributive laws. Given a commuting subset $\{x_j\} \subseteq H$, $y \in H$ and $i \in I$ we need to show that:

$$\pi_i(\bigvee x_j \wedge y) = \pi_i(\bigvee (x_j \wedge y)) \text{ and } \pi_i(y \wedge \bigvee x_j) = \pi_i(\bigvee (y \wedge x_j)) \tag{6.54}$$

First we observe that $\{x_j \wedge y \,|\, j \in J\}$ and $\{y \wedge x_j \,|\, j \in J\}$ are again commuting subsets. Note that if $\pi_i(x_j) \neq 0$ for some j then $\pi_i(\bigvee x_j \wedge y) = \pi(x_j \wedge y) = \pi_i(\bigvee (x_j \wedge y))$. If $\pi_i(x_j) = 0$ for all j then $\pi_i(\bigvee x_j \wedge y) = 0 = \pi_i(\bigvee (x_j \wedge y))$. □

Lemma 6.10. *For every contravariant functor*

$$\mathbf{h} \;:\; \mathcal{C} \longrightarrow \mathsf{cHA} \tag{6.55}$$

and every presheaf $\mathbf{P} \in \mathsf{PSh}(\mathcal{C})$ with a global section $d : \mathbf{1} \longrightarrow \mathbf{P}$ there is a contravariant functor

$$\mathbf{H} \;:\; \mathcal{C} \longrightarrow \mathsf{ncHA} \qquad C \mapsto \mathbf{H}(C) \tag{6.56}$$

where $\mathbf{H}(C)$ *is the complete noncommutative Heyting algebra constructed in the previous lemma from the complete Heyting algebra* $h = \mathbf{h}(C)$ *and the set* $P = \mathbf{P}(C)$*, with presheaf of top elements* $\mathbf{T}$*. Moreover,* $\mathbf{H}/\mathcal{D} \simeq \mathbf{h}$*.*

In the special case when $\mathbf{h} = \mathbf{\Omega}$ *we obtain for every presheaf* $\mathbf{P}$ *with a global section a noncommutative subobject classifier* $\mathbf{H}$ *with* $\mathbf{H}/\mathcal{D} \simeq \mathbf{\Omega}$*.*

Proof. We write down a presheaf version of the construction from Lemma 6.9. The standard way to produce a lattice embedding

$$i \ : \ h \longrightarrow \prod_{i \in I} \mathbf{2} \tag{6.57}$$

is to take $I = \mathbf{PFl}(h)$ to be the set of prime filters on h. We identify $\prod_{i \in I} \mathbf{2}$ with the power set of $\mathbf{PFl}(h)$. For $a \in h$, we have

$$i(a) = \{F \in \mathbf{PFl}(h) \ : \ a \in F\}. \tag{6.58}$$

Now we consider for each object C in $\mathcal{C}$ the complete Heyting algebra $\mathbf{h}(C)$, with $I_C = \mathbf{PFl}(\mathbf{h}(C))$ and $i_C \ : \ \mathbf{h}(C) \longrightarrow \prod_{i \in I_C} \mathbf{2}$ the embedding as above. For each morphism $f : D \to C$ in $\mathcal{C}$, we can construct a commutative diagram

$$\begin{array}{ccccc} \mathbf{h}(C) & \longrightarrow & \prod_{i \in I_C} \mathbf{2} & \longleftarrow & \prod_{i \in I_C} \widehat{\mathbf{P}(C)} \\ \Big\downarrow{\scriptstyle f^*} & & \Big\downarrow{\scriptstyle \alpha_f} & & \Big\downarrow{\scriptstyle \beta_f} \\ \mathbf{h}(D) & \longrightarrow & \prod_{i \in I_D} \mathbf{2} & \longleftarrow & \prod_{i \in I_D} \widehat{\mathbf{P}(D)}. \end{array} \tag{6.59}$$

Here α_f is defined as

$$\alpha_f(S) = \{F \in I_D : \mathbf{h}(f)^{-1}(F) \in I_C\} \tag{6.60}$$

and similarly for β_f. In this way, we get a diagram of presheaves, and we call the pullback of this diagram $\mathbf{H}$. Pullbacks of presheaves are computed pointwise, so $\mathbf{H}(C)$ is a complete noncommutative Heyting algebra for each object C, by Lemma 6.9. Now it is easy to check that the restriction morphisms preserve finite meets and joins, and arbitrary joins for commuting subsets. By Cvetko-Vah [CV19, Theorem 4.5] they then preserve the implication operator as well.

We can define a global section of $\mathbf{H}$ by taking the global section $d : \mathbf{1} \longrightarrow \mathbf{P}$ in each component. Clearly, this global section is contained in the subpresheaf of top elements $\mathbf{T}$. □

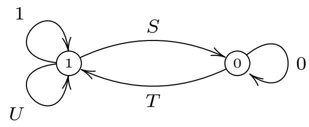
Let us work out an explicit example. Let $\mathcal{C}$ be the category having two objects V and E and two non-identity morphisms $s, t : V \longrightarrow E$, then it is easy to see that the presheaf topos

$$\mathsf{PSh}(\mathcal{C}) \simeq \mathsf{diGraph} \tag{6.61}$$

is the category of directed graphs. A presheaf $\mathbf{P} : \mathcal{C} \longrightarrow \mathbf{Sets}$ determines a set of vertices $\mathbf{P}(V)$ and edges $\mathbf{P}(E)$ and the two maps $\mathbf{P}(s), \mathbf{P}(t) : \mathbf{P}(E) \longrightarrow \mathbf{P}(V)$ assign to an edge its starting resp. terminating vertex. The subobject classifier $\mathbf{\Omega}$ is given by

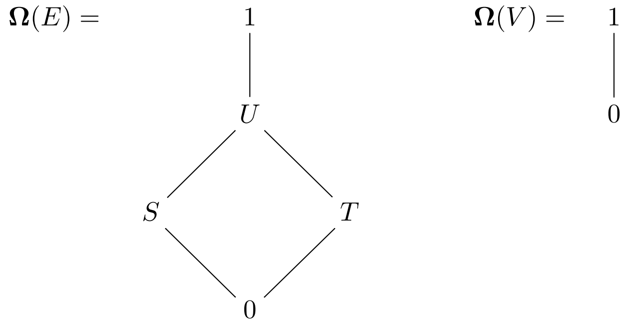
$$\begin{cases} \mathbf{\Omega}(E) = \{1 = \{id_E, s, t\}, U = \{s, t\}, S = \{s\}, T = \{t\}, 0 = \emptyset\} \\ \mathbf{\Omega}(V) = \{1 = \{id_V\}, 0 = \emptyset\} \end{cases} \tag{6.62}$$

and corresponds to the directed graph



(6.63)

with the terminal subobject $\mathbf{1}$ corresponding to the subgraph on the loop 1. The Heyting algebras have poset structure



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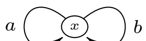
It is easy to verify that there are exactly 4 Lawvere–Tierney topologies on $\mathsf{PSh}(\mathcal{C})$ with corresponding Grothendieck topologies on $\mathcal{C}$ and corresponding sheafifications:

(a) $J_1(V) = \{1\}$ and $J_1(E) = \{1\}$, the chaotic topology. All presheaves are J_1-sheaves and the sheafification functor is the identity.

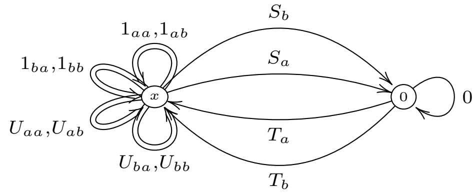
(b) $J_2(V) = \{1\}$ and $J_2(E) = \{1, U\}$. The sheaf condition for $\mathbf{P}$ asserts that for all $v, w \in \mathbf{P}(V)$ there is a unique edge e with $s(e) = v$ and $t(e) = w$. That is, sheaves are the complete directed graphs, and the sheafification of a directed graph is the complete directed graph on the vertices.

(c) $J_3(V) = \{1, 0\}$ and $J_3(E) = \{1\}$. The only non-maximal covering sieve on V is the empty sieve. A presheaf $\mathbf{P}$ is a J_3-sheaf if and only if $\mathbf{P}(V)$ is a singleton. The sheafification sends the vertices of a directed graph all to the same vertex and each edge to a different loop.

(d) $J_4(V) = \{1, 0\}$ and $J_4(E) = \{1, U, S, T, 0\}$, the discrete topology. Here the only sheaf is the terminal object (a one loop graph) and sheafification is the unique map to the terminal object.

Consider the presheaf $\mathbf{P} =$ 

, then the noncommutative subobject classifier $\mathbf{H}$ corresponding to $\mathbf{\Omega}$ and $\mathbf{P}$ as constructed in Lemma 6.10 can be slightly simplified such that $\mathbf{H}(E)$ has only 4 top elements, rather than the 8 given by the construction. The corresponding directed graph is



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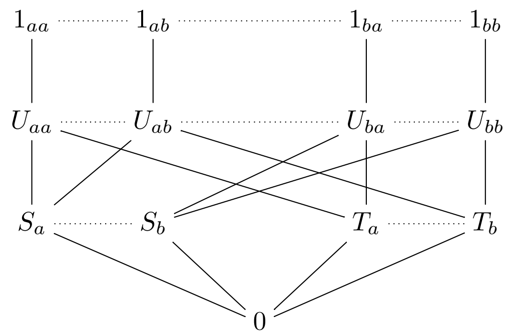
with the subobject $\mathbf{T} \longrightarrow \mathbf{H}$ corresponding to the subgraph on the 4 loops $1_{aa}, 1_{ab}, 1_{ba}$ and 1_{bb}. The poset structure on the noncommutative Heyting algebras is $\mathbf{H}(V) \simeq \mathbf{\Omega}(V) \simeq \mathbf{2}$ and

$\mathbf{H}(E) =$



(6.66)

We will next determine the noncommutative toposes determined by noncommutative Grothendieck topologies associated to $\mathbf{H}$. The category of presheaves is the slice category $\mathsf{PSh}(\mathcal{C})/\mathbf{T}$. A directed graph with a morphism $\pi_{\mathbf{F}} : \mathbf{F} \longrightarrow \mathbf{T}$ is a directed graph with a 4-coloring of its edges. Morphisms in $\mathsf{PSh}(\mathcal{C})/\mathbf{T}$ are directed graph morphisms preserving the coloring of edges.

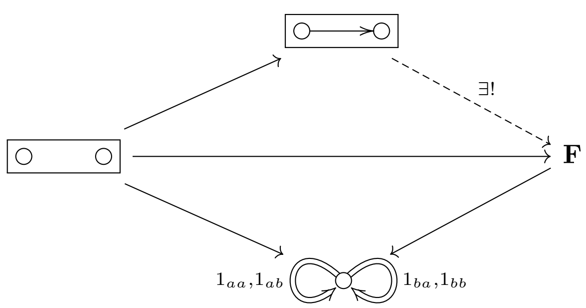
Lemma 6.11. *There are exactly* 16 *noncommutative Grothendieck topologies associated to* $\mathbf{H}$*:*

$$j_{\mathbf{H}}(E) = \{1_{aa}, 1_{ab}, 1_{ba}, 1_{bb}\} \cup S \quad \textit{with} \quad S \subseteq \{U_{aa}, U_{ab}, U_{ba}, U_{bb}\} \tag{6.67}$$

Any 4*-colored digraph satisfies the sheaf condition if* $S = \emptyset$*. For the noncommutative Grothendieck topologies with* $S \neq \emptyset$ *the sheaves are exactly the complete digraphs with a* 4*-coloring.*

Proof. Assume that a noncommutative Lawvere topology $j_{\mathbf{H}} : \mathbf{H} \longrightarrow \mathbf{H}$ is such that $j_{\mathbf{H}}(V)(0) = 1$, then $j_{\mathbf{H}}(E)(0) \in \{1_{aa}, 1_{ab}, 1_{ba}, 1_{bb}, U_{aa}, U_{ab}, U_{ba}, U_{bb}\}$ which is impossible because $j_{\mathbf{H}}(E)$ must be order preserving. Therefore $j_{\mathbf{H}}(V) = id_{\mathbf{H}(V)}$. As a consequence the Grothendieck topologies on $\mathcal{C}$ corresponding to the global sections $1_{aa}, 1_{ab}, 1_{ba}, 1_{bb}$ can only be either J_1 or J_2, giving the 16 cases. If $S = \emptyset$ we have no conditions to satisfy for $\mathbf{F} \longrightarrow \mathbf{T}$.

If, however $S \neq \emptyset$, each occurrence of U_{aa}, U_{ab}, U_{ba} or U_{bb} gives rise to a condition



(6.68)

which means that for every pair of vertices $v, w \in \mathbf{F}(V)$ there must be a unique edge $(v) \longrightarrow (w)$. Note that the color of this unique edge is not imposed by U_{aa}, U_{ab}, U_{ba} or U_{bb}. Therefore, $\mathbf{F}$ is a sheaf for the noncommutative Grothendieck topology if and only if $\mathbf{F}$ is a complete digraph with a certain 4-coloring of the edges determined by the map $\mathbf{F} \longrightarrow \mathbf{T}$. □

It does follow that for any noncommutative Grothendieck topology $J_{\mathbf{H}}$ with $S \neq \emptyset$ the noncommutative topos $\mathsf{Sh}(\mathcal{C}, J_{\mathbf{H}})$ is *not* a Grothendieck topos, nor even an elementary topos, as it fails to have a terminal object (the four loop graph with one loop of each color is *not* a sheaf).

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