Toposes of presheaves on monoids as generalized topological spaces

Toposes online, June 2021

Jens Hemelaer
jens.hemelaer@uantwerpen.be

Universiteit Antwerpen

Based on joint work in progress with Morgan Rogers and on joint work in progress with Aurélien Sagnier.

Section 1

Étale geometric morphisms and complete spreads

(joint work with Morgan Rogers)

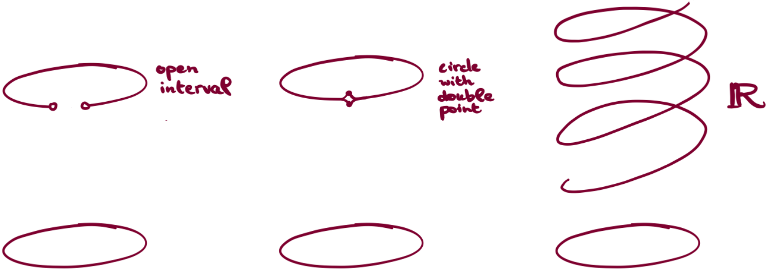
Local homeomorphisms

A sheaf $\mathcal{F}$ on a topological space X can be equivalently described by a local homeomorphism $E \to X$.

Recall that f is a local homeomorphism iff there is an open covering $\{U_i\}_{i \in I}$ of E such that $f|_{U_i}$ is an homeomorphism onto an open set for all $i \in I$.

We use the terminology étale, rather than local homeomorphism, and call E the étale space associated to $\mathcal{F}$.

Local homeomorphisms



Local homeomorphisms

This gives a geometric interpretation to an object of $\mathbf{Sh}(X)$.

We can take this a little further, and think of $\mathcal{F}$ as an étale geometric morphism $\pi : \mathbf{Sh}(E) \to \mathbf{Sh}(X)$.

Here the étale geometric morphisms to $\mathbf{Sh}(X)$ are precisely the geometric morphisms induced by local homeomorphisms.

But there is an alternative characterization that can be extended to other toposes. . .

Fundamental theorem of topos theory

Theorem

If $\mathcal{E}$ is a topos and X is an object in $\mathcal{E}$, then the comma category $\mathcal{E}/X$ is a topos as well.

The category $\mathcal{E}/X$ has as objects the maps $Y \to X$ in $\mathcal{E}$, and as morphisms the morphisms $Y \to Y'$ in $\mathcal{E}$ such that the diagram

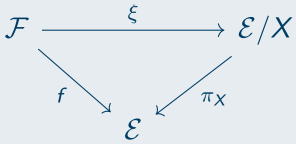
$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

commutes.

Étale geometric morphisms

Definition

A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is étale if and only if there is an object X in $\mathcal{E}$ and an equivalence $\xi : \mathcal{F} \to \mathcal{E}/X$ such that the diagram



commutes, with $\pi_X : \mathcal{E}/X \to \mathcal{E}$ the geometric morphism such that $\pi_X^*(A) \simeq X \times A$.

Étale geometric morphisms

By definition, there is a bijective correspondence

$$\{\text{Objects of } \mathcal{E}\} \leftrightarrow \left\{\text{Étale geometric morphisms } \mathcal{F} \to \mathcal{E}.\right\}$$

This extends to an equivalence of categories:

$$\mathcal{E} \simeq \mathbf{Topos}_{\mathcal{E}}^{\text{ét}}.$$

Presheaves on a monoid

We now look at the topos $\mathbf{PSh}(N)$, the topos of presheaves on the monoid N.

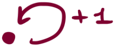
We can identify $\mathbf{PSh}(N)$ with the category of right N-sets.

So étale toposes over $\mathbf{PSh}(N)$ correspond to right N-sets. How can we draw this?

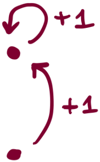
Étale toposes over **PSh**(M)

Take $N = \{0, 1, 2, \dots\}$ under addition.

The base space is the terminal object:



An étale topos over it is for example:



More formally...

Definition (Category of elements)

For N a monoid and X a right N-set, $\int_N X$ is the category with

- as objects the elements of X;
- as morphisms $b \to a$ the elements $n \in N$ such that $an = b$.

So morphisms are of the form $an \xrightarrow{n} a$.
Further, there is a natural projection $\int_N X \to N$.

For every $n \in N$ and every $a \in X$, there is a unique lift $b \xrightarrow{n} a$.
So $\int_N X \longrightarrow N$ is a discrete fibration and every discrete fibration is of this form.

From discrete fibrations to étale maps

It turns out that the étale geometric morphism

$$f : \mathbf{PSh}(N)/X \longrightarrow \mathbf{PSh}(N)$$

agrees with the geometric morphism

$$\mathbf{PSh}(\int_N X) \longrightarrow \mathbf{PSh}(N)$$

induced by the projection functor $\int_N X \longrightarrow N$.

The Root topos

What happens if we take $X = N$, with right N-action given by multiplication?

The Root topos

The Root topos (Connes–Consani, 2019) associated to a monoid N is the topos

$$\mathfrak{Root}(N) \simeq \mathbf{PSh}(\int_N N).$$

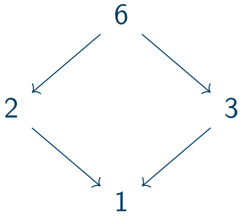
If N is left cancellative, then $\mathfrak{Root}(N)$ is equivalent to a topos of sheaves on a topological space.

The Arithmetic Site

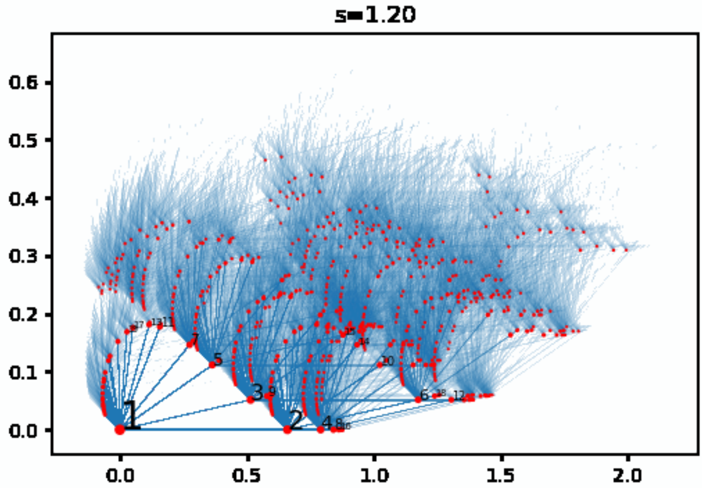
If we take $N = \mathbb{N}_+^\times = \{1, 2, 3, \dots\}$ as monoid under multiplication, then $\mathbf{PSh}(N)$ is the underlying topos of the Arithmetic Site by Connes and Consani.

In this case $\int_N N$ is the category with the nonzero natural numbers as objects, and a unique morphism $m \to n$ whenever n divides m.

This is Conway's big cell, appearing in "Covers of the Arithmetic Site" by Le Bruyn.



Visualization big cell



Wait, what about left N-sets?

Étale geometric morphisms to $\mathbf{PSh}(N)$ correspond to right N-sets.
What about left N-sets?

Is there a dual to the notion of étale geometric morphism?

Answer. Yes! (Bunge–Funk)

Spreads

Let $\mathcal{F}$, $\mathcal{E}$ be (Grothendieck) toposes.

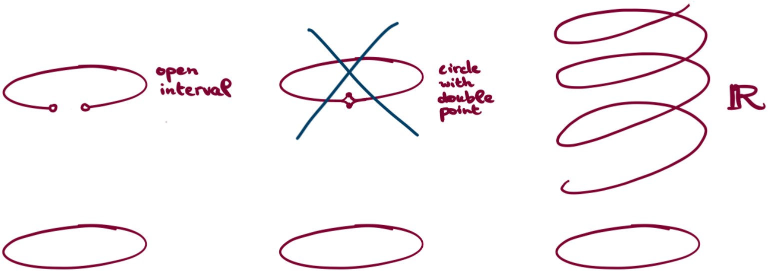
Definition (Bunge–Funk, special case)

A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is a spread if the complemented subobjects of objects of the form $f^*(X)$ give a generating family for $\mathcal{F}$.

This is a generalization of inclusions.

Example. If $Y = \{1, 1/2, 1/3, \dots\} \cup \{0\}$, then the subset $Y \subseteq \mathbb{R}$ induces a spread, for the subspace topology *and* for the discrete topology on Y.

Spreads



Complete geometric morphisms

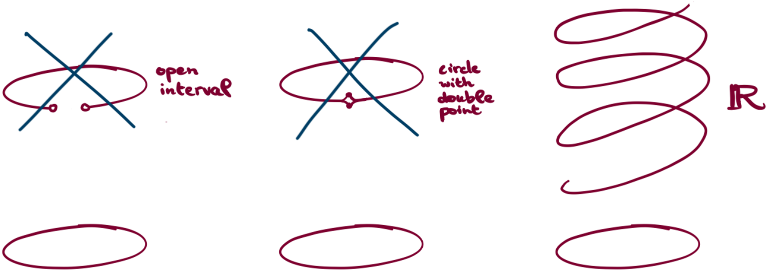
Let $\mathcal{F}$, $\mathcal{E}$ be (Grothendieck) toposes, with $\mathcal{F}$ locally connected.

Definition (Bunge–Funk, special case)

For a geometric morphism $f : \mathcal{F} \to \mathcal{E}$, take a site $(\mathcal{C}, J)$ for $\mathcal{E}$. Consider the category $\mathcal{D}$ with as objects the pairs (C, c) for C in $\mathcal{C}$ and $c \in f^*(C)$ a component, and as morphisms $(C', c') \to (C, c)$ the morphisms $C \to C'$ that respect the choice of component.
We say that f is complete if the following holds:
if $S = \{(C_i, c_i) \to (C, c)\}_{i \in I}$ is a sieve such that the c_i cover c in $\mathcal{F}$, there is a covering sieve R in $\mathcal{C}$ such that $(C', c') \to (C, c)$ is in S for all $a : C' \to C$ in R and all possible choices of c'.

Informally, a covering of $f^{-1}(U)$ comes from a covering on U.

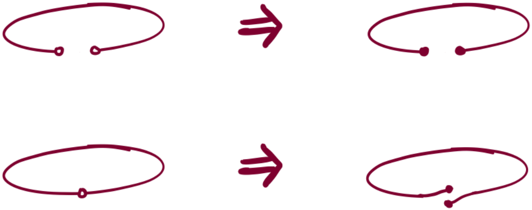
Complete spreads



Spread completion

Theorem (Bunge–Funk)

Every geometric morphism with locally connected domain has a factorization as a pure geometric morphism followed by a complete spread. In particular, a spread factorizes as a pure spread followed by a complete spread.



Distributions

If étale geometric morphisms correspond to objects of the topos, what do complete spreads correspond to?

Definition (Lawvere)

A distribution is a colimit-preserving functor $\mathcal{E} \to \mathbf{Sets}$.

Note that an object of $\mathcal{E}$ is the same as a colimit-preserving functor $\mathbf{Sets} \to \mathcal{E}$, so distributions are dual to objects.

Theorem (Bunge–Funk)

There is a correspondence between distributions on $\mathcal{E}$ and complete spreads with $\mathcal{E}$ as codomain (and locally connected domain).

Distributions for presheaves on a monoid

For $\mathcal{E} = \mathbf{PSh}(\mathcal{C})$, distributions correspond to functors

$$\mathcal{C} \to \mathbf{Sets}.$$

In particular, for $\mathcal{E} = \mathbf{PSh}(M)$, distributions correspond to left M-sets.

What is the associated complete spread?

Distributions for presheaves on a monoid

Definition (Category of elements, dual version)

For N a monoid and Y a left N-set, $\int^N Y$ is the category with

- as objects the elements of Y;
- as morphisms $a \to b$ the elements $n \in N$ such that $na = b$.

So morphisms are of the form $a \xrightarrow{n} na$.
Further, there is a natural projection $\int^N Y \to N$.

For every $n \in N$ and every $a \in X$, there is a unique lift $a \stackrel{n}{\to} na$.
So $\int^N Y \longrightarrow N$ is a discrete opfibration and every discrete opfibration is of this form.

Distributions for presheaves on a monoid

The complete spread associated to the left N-set Y, is then

$$\mathbf{PSh}\left(\int^N Y\right) \longrightarrow \mathbf{PSh}(N)$$

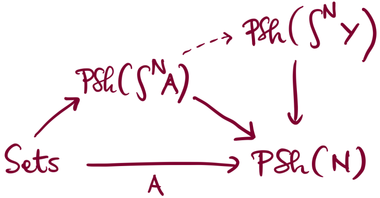
induced by the projection functor $\int^N Y \longrightarrow N$.

Example. We can define a coRoot topos, dual to the Root construction by Connes–Consani.

$$\mathfrak{coRoot}(N) \simeq \mathbf{PSh}\left(\int^N N\right)$$

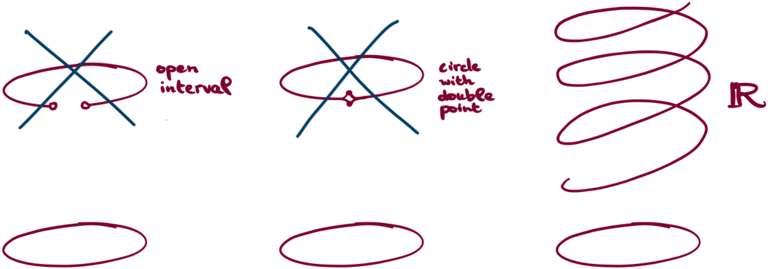
Over-toposes at models

What are the points of $\mathbf{PSh}(\int^N Y)$? Use the factorization:



The points of $\mathbf{PSh}(N)$ are precisely the flat left N-sets. It then follows by the results of Bunge–Funk that the points of $\mathbf{PSh}(\int^N Y)$ are the flat left N-sets together with a morphism of left N-sets $A \to Y$.

Covering maps



Covering maps

In our situation, if a geometric morphism is both a complete spread and étale, then it is a covering map.

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathbf{PSh}(H) \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathbf{PSh}(\pi_1(\mathcal{E})) \end{array}$$

This works for $\mathcal{E} \simeq \mathbf{PSh}(M)$ with M a monoid (Bunge–Funk), or $\mathcal{E} \simeq \mathbf{Sh}(X)$ for X a (Hausdorff, second countable) connected topological manifold (Funk–Tymchatyn).

Fundamental group of a monoid

In $\mathbf{PSh}(M)$, the locally constant objects are precisely the right N-sets on which N acts by bijections.

The full subcategory on the locally constant objects is again a topos, and it is equivalent to $\mathbf{PSh}(G)$ with G the groupification of N.

If X is a locally constant right N-set, then we can define a left N-action on X as well, as multiplication on the right by the inverse. Further, we get

$$\int_N X \simeq \int^N X$$

which shows how the complete spread and étale picture coincide.

Classifying étale geometric morphisms

Let N be a monoid and X a right N-set. Let $\mathbf{PSh}(\int_N X) \longrightarrow \mathbf{PSh}(N)$ be the corresponding étale geometric morphism.

When is $\mathbf{PSh}(\int_N X)$ again of the form $\mathbf{PSh}(M)$ for some monoid M?

Enough that there is an element $x \in X$ such that for each $y \in X$ there is some $u \in N^{\ltimes}$ such that $xu = y$.

In this case $\mathbf{PSh}(\int_N X) \simeq \mathbf{PSh}(N_x)$ for $N_x = \{n \in N : xn = x\}$.

Classifying complete spreads

Let N be a monoid and Y a left N-set. Let $\mathbf{PSh}(\int^N Y) \longrightarrow \mathbf{PSh}(N)$ be the corresponding complete spread.

When is $\mathbf{PSh}(\int^N Y)$ again of the form $\mathbf{PSh}(M)$ for some monoid M?

Enough that there is an element $y \in Y$ such that for each $x \in Y$ there is some $v \in N^{\rtimes}$ such that $y = vx$.

In this case $\mathbf{PSh}(\int^N Y) \simeq \mathbf{PSh}(N^y)$ for $N^y = \{n \in N : ny = y\}$.

Detecting étale geometric morphisms

Theorem (H, Rogers)

Let $\phi : M \to N$ be a monoid map. Then the induced geometric morphism $\mathbf{PSh}(M) \to \mathbf{PSh}(N)$ *is étale if and only if*

1. ϕ *is injective;*
2. *if* $a \in \phi(M)$ *and* $ab \in \phi(M)$ *then* $b \in \phi(M)$*;*
3. *for every* $n \in N$ *there is some* $u \in N^{\ltimes}$ *such that* $nu \in \phi(M)$.

Example. $\phi : \mathbb{N} \to \mathbb{Z}_p, \; n \mapsto p^n$.

Detecting complete spreads

Theorem (H, Rogers)

Let $\phi : M \to N$ be a monoid map. Then the induced geometric morphism $\mathbf{PSh}(M) \to \mathbf{PSh}(N)$ is a complete spread if and only if

1. *ϕ is injective;*
2. *if $b \in \phi(M)$ and $ab \in \phi(M)$ then $a \in \phi(M)$;*
3. *for every $n \in N$ there is some $v \in N^{\rtimes}$ such that $vn \in \phi(M)$.*

Example. $\phi : \mathbb{N} \to \mathbb{Z}_p, \; n \mapsto p^n$.

Section 2

Arithmetic toposes for maximal orders

(joint work with Aurélien Sagnier)

The Arithmetic Site

In their approach to the Riemann Hypothesis, Connes and Consani introduced in 2014 the Arithmetic Site: the topos $\mathbf{PSh}(\mathbb{N}_+^\times)$ equipped with a certain sheaf of semirings.

The sheaf of semirings has an essential role, but the topos $\mathbf{PSh}(\mathbb{N}_+^\times)$ is itself already interesting. For example, it was computed by Connes and Consani that the points of this topos are classified up to isomorphism by the double quotient

$$\mathbb{Q}^*\backslash\mathbb{A}_{\mathbb{Q}}^f/\widehat{\mathbb{Z}}^*$$

with $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ the profinite integers and $\mathbb{A}_{\mathbb{Q}}^f = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$ the finite adeles.

Maximal orders

Take R a Dedekind domain with as field of fractions a global field K.

A maximal order over R is an R-algebra Λ such that Λ is finitely generated torsionfree over R and $\Sigma = \Lambda \otimes_R K$ is a central simple algebra over K.

Examples. $\mathbb{Z}$, $\mathrm{M}_n(\mathbb{Z})$, $\mathbb{F}_p[t]$, $\mathrm{M}_n(\mathbb{F}_p[t])$, $\mathbb{Z}[i]$, $H_{\mathbb{Z}}$, with

$$H_{\mathbb{Z}} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{Z} \text{ or } a, b, c, d \in \tfrac{1}{2} + \mathbb{Z}\}.$$

We write $\Lambda^{\mathrm{ns}} = \Lambda \cap \Sigma^*$.

Maximal orders

Can we associate an Arithmetic Site to maximal orders?

As an underlying topos, is it a good idea to take $\mathbf{PSh}(\Lambda^{\mathrm{ns}})$ or do we need a different topos?

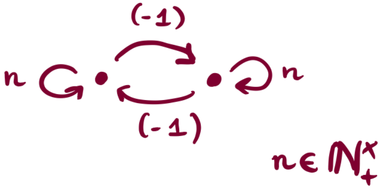
What is the right structure sheaf of semirings? Do we get a spectral interpretation of the zeta function of Λ in topos theoretic terms?

Recovering $\mathbf{PSh}(\mathbb{N}^{\times}_{+})$

If we are being precise, then the Arithmetic Site will not be a special case.

The closest approximation to $\mathbf{PSh}(\mathbb{N}^{\times}_{+})$ is $\mathbf{PSh}(\mathbb{Z}^{\mathrm{ns}})$. How do we recover the topos $\mathbf{PSh}(\mathbb{N}^{\times}_{+})$?

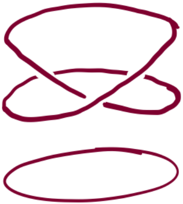
Answer. We take a look at the following $\mathbb{Z}^{\mathrm{ns}}$-set X:



Recovering $\mathbf{PSh}(\mathbb{N}_+^\times)$

For this $\mathbb{Z}^{\mathrm{ns}}$-set, we have $\int_{\mathbb{Z}^{\mathrm{ns}}} X \ \simeq\ \int^{\mathbb{Z}^{\mathrm{ns}}} X \ \simeq\ \mathbb{N}_+^\times$.

We get that $\mathbf{PSh}(\mathbb{N}_+^\times)$ is a two-fold covering space of $\mathbf{PSh}(\mathbb{Z}^{\mathrm{ns}})$.



More covering spaces?

Now take a maximal order Λ and define the profinite elements $\widehat{\Lambda} = \prod_{\mathfrak{p}} \widehat{\Lambda_{\mathfrak{p}}}$, the Λ-adeles $\mathbb{A}_{\Lambda}^{f} = \widehat{\Lambda} \otimes_{R} K$ and the monoid

$$\widehat{\Lambda}^{\mathrm{ns}} = \widehat{\Lambda} \cap (\mathbb{A}_{\Lambda}^{f})^{*}.$$

Then $\mathbf{PSh}(\widehat{\Lambda}^{\mathrm{ns}})$ is a nicely behaved topos: most importantly $\mathfrak{Root}(\widehat{\Lambda}^{\mathrm{ns}})$ is a spectral space (coherent topos). This is not necessarily true for $\mathbf{PSh}(\Lambda^{\mathrm{ns}})$ itself, e.g. for $\Lambda = \mathbb{Z}[\sqrt{-5}]$.

Going from $\widehat{\Lambda}^{\mathrm{ns}}$ to Λ^{ns}

To go from $\widehat{\Lambda}^{\mathrm{ns}}$ to Λ^{ns} we look at the right $\widehat{\Lambda}^{\mathrm{ns}}$-set

$$\Sigma^*\backslash(\mathbb{A}_\Sigma^f)^*$$

with the $\widehat{\Lambda}^{\mathrm{ns}}$-action given by multiplication on the right.

The action is by bijections, so we also have a left $\widehat{\Lambda}^{\mathrm{ns}}$-action, defined as multiplication on the right by the inverse.

So we are in the situation where we have a covering map (both complete spread and étale). In symbols

$$\int_{\widehat{\Lambda}^{\mathrm{ns}}} \Sigma^*\backslash(\mathbb{A}_\Sigma^f)^* \simeq \int^{\widehat{\Lambda}^{\mathrm{ns}}} \Sigma^*\backslash(\mathbb{A}_\Sigma^f)^*.$$

Can we give a more concrete description?

Going from $\widehat{\Lambda}^{\mathrm{ns}}$ to Λ^{ns}

In the category of elements, two objects are isomorphic if they are related by a unit, i.e. by an element of $\widehat{\Lambda}^*$.

So the objects of the category of elements are classified up to isomorphism by the double quotient

$$\Sigma^* \backslash (\mathbb{A}_\Sigma^f)^* / \widehat{\Lambda}^*.$$

which is known as the ideal class group.

There is a special element $[1] \in \Sigma^* \backslash (\mathbb{A}_\Sigma^f)^*$, that has as endomorphism monoid exactly Λ^{ns}. The other elements may have endomorphism monoid $(\Lambda')^{\mathrm{ns}}$ for a different maximal order Λ'.

The full ring of adeles

We now restrict to the case $\Lambda = \mathbb{Z}$.

The points of $\mathbf{PSh}(\mathbb{Z}^{\mathrm{ns}})$ are classified up to isomorphism by the double quotient

$$\mathbb{Q}^* \backslash \mathbb{A}_{\mathbb{Q}}^f / \widehat{\mathbb{Z}}^*.$$

Is there an alternative topos where $\mathbb{A}_{\mathbb{Q}}^f$ gets replaced by the full ring of adeles $\mathbb{A}_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q}}^f \times \mathbb{R}$ here?

Answer. Yes, see the scaling site by Connes–Consani (2015).

The scaling site takes the topology on $\mathbb{R}$ into account. We give an alternative construction where we equip with $\mathbb{R}$ with the discrete topology.

The full ring of adeles

Consider $\mathbb{R}$ as a $\mathbb{Z}^{\mathrm{ns}}$-set under multiplication, and the covering map

$$\mathbf{PSh}(\int^{\mathbb{Z}^{\mathrm{ns}}} \mathbb{R}) \longrightarrow \mathbf{PSh}(\mathbb{Z}^{\mathrm{ns}}).$$

The points of $\mathbf{PSh}(\int^{\mathbb{Z}^{\mathrm{ns}}} \mathbb{R})$ are given by flat left $\mathbb{Z}^{\mathrm{ns}}$-sets A together with a map

$$A \longrightarrow \mathbb{R}$$

of left $\mathbb{Z}^{\mathrm{ns}}$-sets. As a result, the points are classified up to isomorphism by

$$\mathbb{Q}^* \backslash \mathbb{A}_{\mathbb{Q}} / \widehat{\mathbb{Z}}^*.$$

The full ring of adeles

We can decompose $\mathbb{R}$ as $\mathbb{R}^* \sqcup \{0\}$, and the $\mathbb{Z}^{\mathrm{ns}}$-action on $\{0\}$ is trivial, so $\{0\}$ is a copy of the terminal object.

This means that $\mathbf{PSh}(\int^{\mathbb{Z}^{\mathrm{ns}}} \mathbb{R})$ contains a copy of $\mathbf{PSh}(\mathbb{Z}^{\mathrm{ns}})$.

This corresponds to the decomposition

$$\mathbb{Q}^* \backslash \mathbb{A}_{\mathbb{Q}} / \widehat{\mathbb{Z}}^* = \left(\mathbb{Q}^* \backslash (\mathbb{A}_{\mathbb{Q}}^f \times \{0\}) / \widehat{\mathbb{Z}}^* \right) \sqcup \left(\mathbb{Q}^* \backslash (\mathbb{A}_{\mathbb{Q}}^f \times \mathbb{R}^*) / \widehat{\mathbb{Z}}^* \right)$$

considered by Connes–Consani for the scaling site.

The full ring of adeles

More precisely, there is a decomposition

$$\mathbf{PSh}\left(\int^{\mathbb{Z}^{\mathrm{ns}}} \mathbb{R}\right) = \mathbf{PSh}(\mathbb{Z}^{\mathrm{ns}}) \sqcup \bigsqcup_{a \in \mathbb{R}^*/\mathbb{Q}^*} \mathbf{PSh}\left(\int^{\mathbb{Z}^{\mathrm{ns}}} \mathbb{Q}^*\right).$$

This hopefully gives a simplified picture of what is happening with the scaling site, and some intuition about how covering maps work in practice.